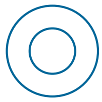
Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 6, November 25, 2022

Quick check A (medial axis, condition number). Compute the medial axis and the condition number of the subset $S^1 \cup 2 \cdot S^1 \subset \mathbb{R}^2$ (with respect to the Euclidean metric).



Quick check B (components). Compute the connected components of the simplicial complex $\{\emptyset, \{0\}, \dots, \{5\}, \{0,2\}, \{0,3\}, \{0,5\}, \{1,4\}, \{2,5\}, \{0,2,5\}\}$ via the algorithm discussed in the lecture. Illustrate!

Quick check C (homological algebra). Refresh your memory on chain complexes, homology, chain maps, and chain homotopy. We will briefly recall these terms in the lectures, but this is an additional opportunity to ask questions.

Exercise 1 (condition numbers; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If M, N are closed smooth submanifolds of $\mathbb{R}^2$ with $M \cap N = \emptyset$, the the condition number of $M \cup N$ is the minimum of the condition numbers of M and N.

Exercise 2 (number of components, algorithmically; 3 credits). Provide an algorithm that, given a finite simplicial complex X, computes the number of connected components of X. Explain your implementation model of simplicial complexes and prove that your algorithm is correct.
Hints. You may base this on algorithmic material from the lecture or start from scratch.

Exercise 3 (degrees of vertices, algorithmically; 3 credits). Provide an algorithm that, given a finite simplicial complex X and a vertex x of X computes the degree of x in the graph determined by the vertices and 1-simplices of X. Explain your implementation model of simplicial complexes and prove that your algorithm is correct.

Exercise 4 (null-homotopic maps of spheres; 3 credits). Let $m, n \in \mathbb{N}$ with $m < n$. Show that every continuous map $S^m \longrightarrow S^n$ is null-homotopic.
Hints. Use ? to show that every continuous map $S^m \longrightarrow S^n$ is homotopic to a map that is not ??. Then apply the ??? to conclude.

Bonus problem (implementation; 3 credits each). Implement these algorithms in your favourite programming language (document your code appropriately!):

1. Computation of connected components of finite simplicial complexes, using the union-find framework.
2. Computation of the number of connected components of finite simplicial complexes as in Exercise 2.
3. Computation of degrees of vertices in finite simplicial complexes as in Exercise 3.

Submission before December 2, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on December 1, 2022.