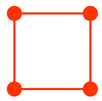
Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 7, December 2, 2022

Quick check A (simplicial homology of simplicial intervals). Compute $H_n([0, N]_\Delta)$ for all $n, N \in \mathbb{N}$.

Quick check B (simplicial homology of the hollow square). Compute the simplicial homology (in all degrees) of the following simplicial complex:



Quick check C (simplicial homology of not so hollow squares). Compute the simplicial homology (in all degrees) of the following simplicial complexes:

Exercise 1 (trivial simplicial homology; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> There exists an infinite simplicial complex X that satisfies $H_n(X) \cong_\mathbb{Z} 0$ for all $n \in \mathbb{N}_{>0}$.

Exercise 2 (simplicial homology of the simplicial 2-sphere; 3 credits). Compute $H_n(S(2))$ for all $n \in \mathbb{N}$.

Exercise 3 (simplicial homology of the simplicial Möbius strip; 3 credits). Give a triangulation of the (closed) Möbius strip and compute the simplicial homology of this simplicial complex. Illustrate!

Exercise 4 (simplicial homology in degree 0; 3 credits). Let X be a finite simplicial complex with m connected components. Show that $H_0(X) \cong_\mathbb{Z} \mathbb{Z}^m$.

Bonus problem (custom-made simplicial complex; 3 credits). Give an example of a finite simplicial complex X with

$$H_0(X) \cong_\mathbb{Z} \mathbb{Z} \quad \text{and} \quad H_1(X) \cong_\mathbb{Z} \mathbb{Z}/2022 \quad \text{and} \quad H_{2022}(X) \cong_\mathbb{Z} \mathbb{Z}.$$

Prove that your example does have this property!

Bonus problem (Nikolausaufgabe; 3 credits). The Blorx Building Trust once more has won the bid to construct the *Haus des Nikolaus*. The construction turned out to be cheap, but also somewhat lacking: Blorx only delivered the set

$$\big\{(0,0), (1,0), (1,1), (0,1), (0.5, 2)\big\} \subset (\mathbb{R}^2, d_2)$$

of vertices, the brand-new Čech-Complex-Constructor$^{\mathrm{TM}}$, and the instruction: Setting the radius appropriately will maximise your homological experience!

Which radii lead to the biggest (in terms of rank) simplicial homology of the corresponding Čech complexes in degree 1 ? Justify your answer!

Submission before December 9, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on December 8, 2022.