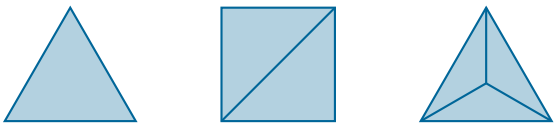
Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 8, December 9, 2022

The Python library simplicial ("Simplicial topology in Python") and its documentation are available at https://simplicial.readthedocs.io/en/latest/index.html.

Quick check A (Mayer–Vietoris sequence)**.** Compute $H_n(S(2))$ for all $n \in \mathbb{N}$ inductively via the Mayer–Vietoris sequence, starting from a 2-simplex and then adding one 2-simplex at a time.



$S(2)$

Quick check B (Smith normal form)**.** Compute the Smith normal form (over $\mathbb{Z}$) of the following matrix:

$$\begin{pmatrix} 2 & 3 & 7 & 1 \\ 3 & 1 & -2 & 1 \\ 1 & 0 & 2 & 3 \end{pmatrix}$$

Quick check C (algorithmic computation of simplicial homology)**.** Compute the $\mathbb{F}_2$-Betti numbers of the simplicial complexes on Sheet 7, using the simplicial library for Python.

Exercise 1 (simplicial homology of unions; 3 credits)**.** Is the following statement true? Justify your answer with a suitable proof or counterexample.

If X and Y are finite simplicial complexes, then

$$b_2(X \cup Y; \mathbb{Z}) = b_2(X; \mathbb{Z}) + b_2(Y; \mathbb{Z}) - b_2(X \cap Y; \mathbb{Z}).$$

Exercise 2 (homology of simplicial spheres; 3 credits)**.** Let $d \in \mathbb{N}_{>0}$. Compute the homology $H_n(\Delta(d+1), S(d))$ for all $n \in \mathbb{N}$ directly from the definition. Use this result to compute $H_n(S(d))$ for all $n \in \mathbb{N}$.

Exercise 3 (an alternating sum of binomial coefficients via homology; 3 credits)**.** Let $d \in \mathbb{N}$. Use the computation of $H_n(\Delta(d))$ for all $n \in \mathbb{N}$ to show that

$$\sum_{k=1}^{d+1} (-1)^{k-1} \cdot \binom{d+1}{k} = 1,$$

Exercise 4 (barycentric subdivision in the Python library simplicial; 3 credits)**.** What does the method SimplicialComplex.barycentricSubdivide(simplex) from the Python library simplicial do? Give a mathematical definition of this subdivision. Illustrate!

Bonus problem (simplicial products in Python; 3 credits)**.** Write a Python method that computes the simplicial product of two finite simplicial complexes. Use this method to compute the $\mathbb{F}_2$-Betti numbers of $S(1) \boxtimes \Delta(0)$, $S(1) \boxtimes \Delta(1)$, $S(1) \boxtimes S(1)$, and $S(1) \boxtimes S(2)$. Document your code!

Submission before December 16, 2022, 8:30, via GRIPS (in English or German)
The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on December 15, 2022.