Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 9, December 16, 2022

Quick check A (simplicial homology of continuous maps)**.** Let X be a finite simplicial complex and let $n \in \mathbb{N}$. Determine $H_n(\beta_X)\colon H_n(\operatorname{sd} X) \longrightarrow H_n(X)$.

Quick check B (self-maps of the circle I)**.** Let $\varphi\colon |S(1)| \longrightarrow |S(1)|$ be the map "wrapping the simplicial circle around itself twice". Give a precise description of φ and compute $H_1(\varphi)\colon H_1(S(1)) \longrightarrow H_1(S(1))$.

Quick check C (self-maps of the circle II)**.** Does every continuous map $S^1 \longrightarrow S^1$ have a fixed point?

Quick check D (ƨtɘhɔƨɟɘ⅃?)**.** Reflect about the question whether there exists a finite simplicial complex X and a continuous map $\varphi\colon |X| \longrightarrow |X|$ that has a fixed point and satisfies $\Lambda(\varphi; K) = 0$ for all fields K.

Exercise 1 (self-maps of spheres; 3 credits)**.** Is the following statement true? Justify your answer with a suitable proof or counterexample.

If $f\colon S^{2022} \longrightarrow S^{2022}$ is a continuous map, then f^{2022} has a fixed point.

Exercise 2 (Lefschetz number of chain maps; 3 credits)**.** Let K be a field and let C be a chain complex over K that has only finitely many non-zero chain modules and such that each chain module is finite-dimensional. Let $f\colon C \longrightarrow C$ be a chain map. Show that

$$\sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr} H_n(f) = \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr} f_n.$$

Hints. Choose convenient bases.

Exercise 3 (Sperner's lemma; 3 credits)**.** Derive the classical version of Sperner's lemma from the manifold version.

Hints. Show the case of dimension 1 by hand. Then, proceed by induction. You may use without proof: If (X, φ) is a subdivision of $\Delta(n)$, then X is an n-pseudomanifold with boundary.

Exercise 4 (piece of cake; 3 credits)**.** What is the Simmons–Su protocol for envy-free cake division? Explain the terminology, the protocol, and the role of Sperner's lemma.

Hints. Don't forget to cite your sources!

Bonus problem (graph-theoretic proof of Sperner's lemma; 3 credits)**.** Give a homology-free proof of Sperner's lemma for manifolds by considering the dual graph (plus an "external" vertex) and shaking hands.

Submission before December 23, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on December 22, 2022.