Seminar: CAT(0) Cube Complexes

K. Li/C. Löh/M. Uschold February 2023

A metric space is CAT(0) if geodesic triangles are "slimmer" than in Euclidean space. CAT(0) spaces have many nice geometric properties, e.g., they admit projections onto convex subsets. While it is difficult to verify if a metric space is CAT(0) in general, it is surprisingly easy for the class of cube complexes. Cube complexes are spaces obtained by gluing cubes (of possibly different dimensions) along their faces. A first goal of the seminar will be to prove Gromov's link condition, which characterises when cube complexes are CAT(0).

If a group acts on a CAT(0) cube complex, this has strong consequences, e.g., the group satisfies the Tits alternative. A second goal will be the show that Coxeter groups, which can be thought of as reflection groups, act on CAT(0) cube complexes. To this end we will develop a convenient construction of CAT(0) cube complexes from a system of halfspaces.

References. The seminar structure follows the lecture notes of Schwer [Sch]. However, these should be read with great care due to typos. It is indispensable to consult other references as well [BH99, Wil, Dav08].

Prerequisites. Basic knowledge about metric spaces and groups will be assumed.

Admin and preparation. Please take the general advice on seminars into account: https://loeh.app.uni-regensburg.de/teaching/seminar_preparation.pdf

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CAT(κ)-Geometry

Talk 1 (The model spaces).
Main reference: [BH99, Chapter I.6]

In Talk 2, we will define CAT(κ)-spaces by comparing triangles to triangles in the model spaces. We introduce these model spaces $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$. We focus on the hyperbolic space $\mathbb{H}^n$ and recall/prove the following:

- Recall the necessary basic notions of metric geometry [BH99, Section I.1].
- Introduce the Poincaré Ball model and the halfspace model for $\mathbb{H}^n$.
- State [BH99, Proposition I.6.8, I.6.10, I.6.11] and [BH99, Proposition I.6.13] and prove these in one of the models.
- Introduce the Möbius group.
- Briefly mention the Klein model.

Talk 2 (Basic notions of CAT(κ)-metric spaces).
Main reference: [Sch, Section 1], [BH99, Chapter II.1], [Wil, Section 2.1]

A metric space is CAT(κ) if geodesic triangles are "slimmer" than in the corresponding model space ($\mathbb{S}^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$). We show basic properties of these spaces and prove that hyperbolic spaces satisfy the CAT(0)-condition.

- Introduce the CAT(κ)-condition and CAT(κ)-spaces.
- Basic properties of CAT(κ)-spaces [Sch, until Proposition 1.10].
- Examples: Recall the 2-dimensional model spaces $\mathbb{S}^2$, $\mathbb{E}^2$, and $\mathbb{H}^2$, which were introduced in Talk 1.
- Prove that $\mathbb{H}^n$ is a CAT(0)-space [BH99, Theorem II.1.12].

Talk 3 (Results from CAT(0)-geometry).
Main reference: [Sch, Section 1], [BH99, Chapter II.1, II.4]

We focus on the case $\kappa = 0$, and state some results on CAT(0)-spaces.

- State the Cartan–Hadamard Theorem [Sch, Theorem 1.11].
- Prove the statements on projections onto convex sets [Sch, Proposition 1.13] and fixed-point sets [Sch, Proposition 1.15]
- Prove that CAT(0)-spaces are contractible [BH99, Corollary II.1.5]. As a consequence, if a group acts 'nicely' on a CAT(0)-space, the quotient is a classifying space for the acting group.

Cube complexes

Talk 4 (Cube complexes).
Main reference: [Sch, Section 2], [BH99, Definition I.7.32]

Cube complexes are a method for building metric spaces that have a combinatorial flavour.

- Define cube complexes.
- Introduce the metric on cube complexes.
- Treat Examples [Sch, Example 2.4] and [Sch, Example 2.6].
- Prove the proposition on string lengths [Sch, Proposition 2.8].
- Prove the proposition on when a cube complex is a complete metric space [Sch, Proposition 2.11].
- Time permitting: State the counterexample [Sch, Example 2.12].

Talk 5 (Gromov's link condition).
Main reference: [BH99, Section II.5], [Sch, Section 2]

In general, it is hard to check the CAT(κ)-condition introduced in Talk 2. For cube complexes, Gromov's link condition provides an accessible criterion to when a cube complexes is a CAT(0)-metric space.

- Introduce Gromov's link condition.
- Prove the case $\kappa = 0$, i.e., the characterisation of CAT(0)-cube complexes [BH99, Theorem II.5.2] [Sch, Theorem 2.18].
- State and prove [BH99, Theorem II.5.4] in the case $\kappa = 0$.
- Discuss the characterisation in dimension 2 [BH99, Lemma II.5.6].

Talk 6 (Right-angled Artin groups).
Main reference: [Wil, Section 3.4], [Sch, Example 2.23], [Cha07, Section 2.6]

Right-angled Artin groups (RAAGs) are an easy to define yet rich class of groups. There is a canonical action of a RAAG on a cube complex, which we show to be CAT(0) using the link condition from Talk 5.

- Define right-angled Artin groups and discuss examples.
- Show that the class of RAAGs is closed under direct products, free products, and more generally amalgamated products over special subgroups.
- Construct the Salvetti complex and show that its universal covering satisfies the CAT(0) condition.

Hyperplanes and halfspace systems

Talk 7 (Hyperplanes).
Main reference: [Sch, Section 3], [Sag14, Lecture 1, Section 2]

Cube complexes have hyperplanes, which have interesting properties in the CAT(0) setting: A hyperplane divides the cube complex into two halfspaces. Moreover, a family of hyperplanes that intersect pairwise has a non-empty joint intersection.

- Introduce the notion of hyperplanes.
- Prove [Sch, Proposition 3.4].
- Prove [Sch, Proposition 3.6].

Talk 8 (Cube complexes from halfspace systems: construction).
Main reference: [Sch, Section 4]

An abstract system of halfspaces determines a CAT(0) cube complex. This construction will be used in Talk 12 to show that Coxeter groups act on CAT(0) cube complexes.

- Define halfspace systems.
- State [Sch, Theorem 4.3].
- Explain how to construct a cube complex from a halfspace system.
- Prove [Sch, Lemma 4.7 and Lemma 4.9].

Talk 9 (Cube complexes from halfspace systems: CAT(0) property).
Main reference: [Sch, Section 4]

We show that the cube complex constructed from a halfspace system is CAT(0).

- Recall the construction of a cube complex from a halfspace system from Talk 8.
- Prove [Sch, Lemma 4.10].
- Prove [Sch, Theorem 4.3].

Talk 10 (Median graphs).
Main reference: [Hag, Chapter 1], [Gen, p. 6, Chapter I.3]

In Talk 8 and 9, we saw that CAT(0)-cube complexes are determined by halfspace systems. In this talk, we introduce another combinatorial viewpoint and show its equivalence.

- Define median graphs and hyperplanes in median graphs [Hag, Section 1.3].
- Give examples.
- Show that the 'median graph' and 'CAT(0)-cube complex' viewpoints are equivalent [Hag, Theorem 1.17, Lemma 1.20].

Cubulating Coxeter groups

Talk 11 (Coxeter groups).
Main reference: [Sch, Sections 5–6], [Dav08, Chapters 2–3]

Coxeter groups are an important class of groups in geometric group theory. We discuss some examples and basic properties.

- Define Caley graphs and word metrics.
- Introduce Coxeter groups and discuss examples.
- Prove [Sch, Lemma 6.8].

Talk 12 (Cubulating Coxeter groups).
Main reference: [Sch, Section 6]

We construct a halfspace system associated to a Coxeter group. It follows that Coxeter groups act on CAT(0) cube complexes.

- Explain the notion of walls in 'the' Cayley graph.
- Prove [Sch, Lemma 6.10 and Lemma 6.12].
- Prove [Sch, Proposition 6.14].
- Prove [Sch, Proposition 6.16].
- Discuss [Sch, Example 6.18 and Example 6.19].
- State [Sch, Open Problem 6.21].

Applications

Talk 13 (Consequences of cubulation).
Main reference: [Sch, Section 7], [Hag, Section 3.1]

This talk has mostly the character of a survey. Admitting a geometric action on a CAT(0) cube complex has strong consequence for the structure of a group. For example, the group satisfies the Tits alternative, which is a drastic dichotomy for its subgroups.

- Discuss the Tits alternative and outline the proof of [Sch, Theorem 7.27].
- Choose and present some other consequences of cubulation [Hag, Section 3.1].

Talk 14 (Virtual Haken conjecture and fibration theorems).
Main reference: [Cal], [Ago13]

In 2012, Ian Agol proved the Virtual Haken conjecture, which states that every aspherical, closed 3-manifold has a finite cover that is Haken. Surprisingly, this major conjecture in 3-manifold topology was solved using actions on CAT(0)-cube complexes. This talk should be more of a survey talk, outlining the main ideas.

- State the Virtual Haken conjecture (VHC) and the virtual fibring conjecture (VFC).
- Explain the necessary notions from 3-manifold topology.
- State the version of Agol's theorem [Cal, Theorem 7.1] and explain how this implies the VHC.
- Outline the main ideas in the proof of Agol's theorem.

References

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[Cha07] Ruth Charney. An introduction to right-angled Artin groups. *Geom. Dedicata*, 125:141–158, 2007.

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