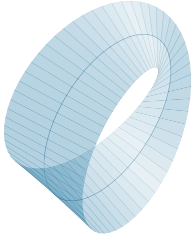
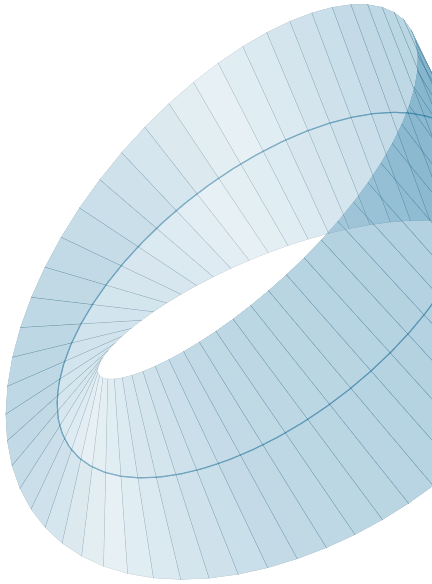


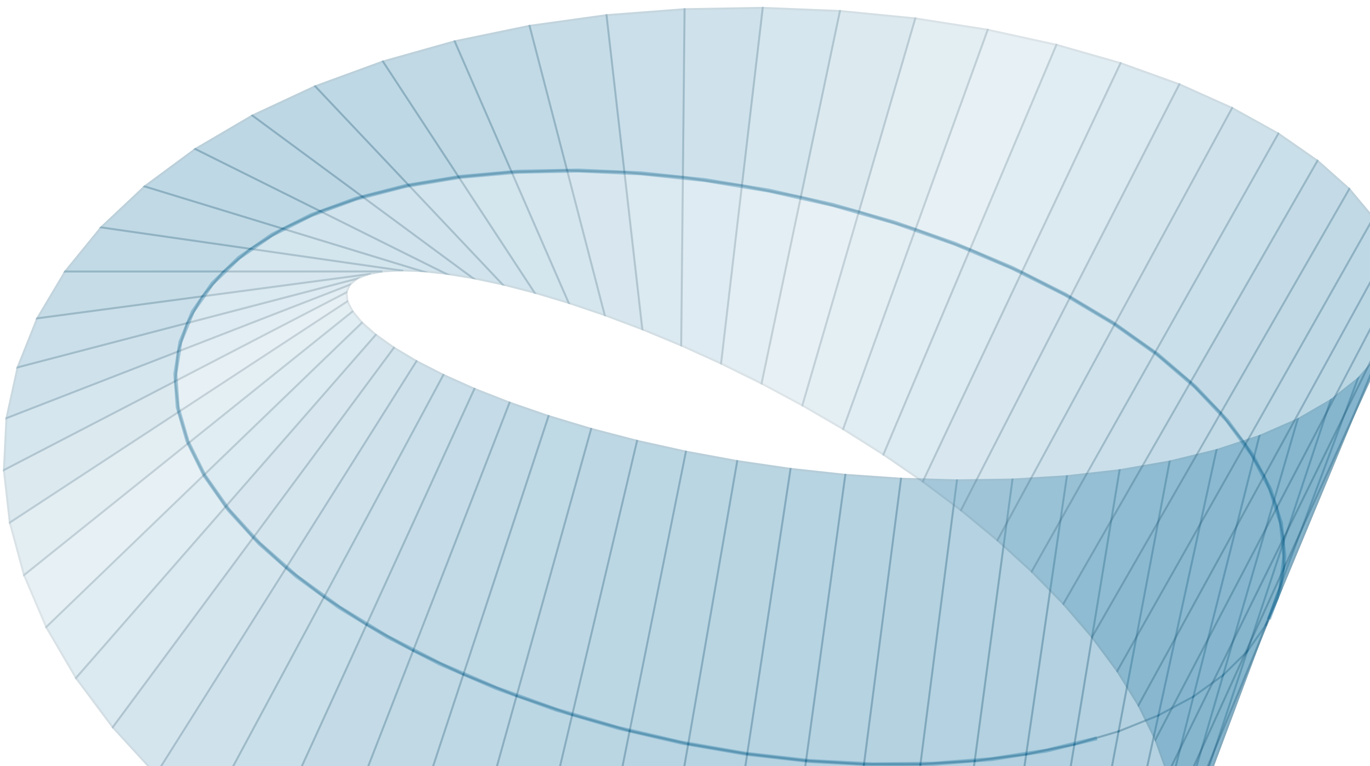


Differential Geometry I

Wintersemester 2020/21

Universität Regensburg

Clara Löh



Version of February 11, 2021
clara.loeh@mathematik.uni-regensburg.de
Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg

Contents

Guide to the literature

This course will not follow a single source (but several parts are based on Lee's book *Riemannian manifolds*) and there are many books that cover the standard topics (all with their own advantages and disadvantages). Therefore, you should individually compose your own favourite selection of books.

Differential/Riemannian geometry

- C. Bär. *Differential Geometry*, lecture notes, summer term 2013, Universität Potsdam.
  https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/skript-DiffGeo-engl.pdf
- J. Cheeger, D. G. Ebin. *Comparison theorems in Riemannian geometry*, revised reprint of the 1975 original, AMS, 2008.
- S. Kobayashi, K. Nomizu. *Foundations of differential geometry. Vol. I*, reprint of the 1963 original, Wiley Classics Library, John Wiley & Sons, 1996.
- S. Kobayashi, K. Nomizu. *Foundations of differential geometry. Vol. II*, reprint of the 1969 original, Wiley Classics Library, John Wiley & Sons, 1996.
- J. M. Lee. *Riemannian manifolds. An introduction to curvature*, Graduate Texts in Mathematics, 176, Springer, 1997.
  A quick, concise, and geometric introduction; the second edition contains more material, in particular, more advanced applications.
- J. M. Lee. *Introduction to Riemannian manifolds*, second edition, Graduate Texts in Mathematics, 176, Springer, 2018.

- P. Petersen. *Riemannian geometry*, third edition, Graduate Texts in Mathematics, 171, Springer, 2016.

Background: Analysis on manifolds and topology

- B. Ammann. *Analysis auf Mannigfaltigkeiten*, lecture notes, summer term 2020, Universität Regensburg.
  http://www.mathematik.uni-regensburg.de/ammann/lehre/2020s_analysisIV/AnalysisIV-V2020-10-26.pdf
- J. M. Lee. *Introduction to smooth manifolds*, second edition, Graduate Texts in Mathematics, 218, Springer, 2013.
- C. Löh. *Algebraic Toplogy*, lecture notes, winter term 2018/19, Universität Regensburg.
  http://www.mathematik.uni-regensburg.de/loeh/teaching/topologie1_ws1819/lecture_notes.pdf
- W. S. Massey, *Algebraic topology: an introduction*, reprint of the 1967 edition, Graduate Texts in Mathematics, 56, Springer, 1977.
- J. R. Munkres. *Topology*, second edition, Prentice Hall, 2000.

Further topics and related fields

- M. Berger. *A panoramic view of Riemannian geometry*, Springer, 2003.
- M. R. Bridson, A. Haefliger. *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, 319, Springer, 1999.
- C. Druţu, M. Kapovich. *Geometric Group Theory*, Colloquium Publications, 63, American Mathematical Society, 2018.
- M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, based on the 1981 French original, with appendices by M. Katz, P. Pansu and S. Semmes, translated from the French by S. M. Bates, Progress in Mathematics, 152, Birkhäuser, 1999.
- C. Löh. *Geometric Group Theory. An Introduction*, Universitext, Springer, 2018.
  Errata: http://www.mathematik.uni-r.de/loeh/ggt_book/errata.pdf

0

Introduction

What is differential geometry?

Differential geometry is the study of geometric objects by analytic means. Geometric objects in this context usually are Riemannian manifolds, i.e., smooth manifolds with a Riemannian metric. This allows to define lengths, volumes, angles, ... in the language of (multilinear) analysis. A central concern of differential geometry is to define and compare (intrinsic) notions of curvature of spaces. A particularly fascinating aspect is that many local curvature constraints are reflected in the global shape.

Differential geometry has various applications in the formalisation of Physics, in medical imaging, and also in other fields of theoretical mathematics. For example, certain phenomena in group theory and topology can only be understood via the underlying geometry.

In this course, we will introduce basic notions of differential geometry. This includes, in particular, Riemannian metrics, the Riemannian curvature tensor, sectional curvature, Ricci curvature, and scalar curvature. Moreover, we will study some first global obstructions to curvature constraints.

Why differential geometry?

Differential geometry serves both as a computational tool and as a language to model various types of situations.

Classical applications of differential geometry include the following:

Geometry and topology

- Proof of the Poincaré conjecture

  Question. How can the 3-sphere be characterised?

- Rigidity phenomena of manifolds
- (Alternative approaches to) computation of cohomological invariants
- ...

Group theory and other algebraic fields

- Growth of groups

  Question. Let Γ be a finitely generated group and let $S \subset \Gamma$ be a finite generating set. What can be said about the asymptotic behaviour of the sequence $(|S^n|)_{n\in\mathbb{N}}$?

- Rigidity phenomena of groups
- Approximation results for cohomological invariants of groups
- Representation theory of infinite groups
- ...

Physics

- Modelling mechanical systems
- Modelling fluids
- Modelling space-time (and formulating general relativity)
- ...

Real world

- Cartography

  Question. Is it possible to draw length-preserving planar maps of parts of the Earth? What about angle- or area-preserving maps?

- Modelling human physiology and anatomy
- Simulating non-Euclidean (e.g., spherical!) worlds in computer games.
- ...

In addition to these concrete applications, differential geometry also had substantial influence on the creation of new fields: For example, systematically integrating the (differential) geometric point of view into group theory led to *Geometric group theory* (which then in turn had a big impact on differential geometry) [12, 5, 14, 8, 19].

The Poincaré conjecture

As a striking example, we briefly sketch how differential geometry solves a purely topological problem:

Theorem (Poincaré conjecture [27, 28, 29, 16, 26]). *Every simply connected compact* 3*-manifold without boundary is homeomorphic to the* 3*-dimensional sphere* $\mathbb{S}^3$.

The statement does not make any reference to differential geometry; so, how can differential geometry be used in the proof? Let M be a simply connected compact 3-manifold without boundary (*simply connected* means that M is path-connected and that every continuous loop in M can be continuously deformed into a constant map; i.e., M does not have "holes" that could be "surrounded" by a loop). So far, M is only a topological object. One then adds geometric structure to M:

- It is known that every 3-manifold admits an essentially unique smooth structure; thus, we equip M with such a smooth structure and view M as s smooth manifold.
- Every smooth manifold admits a Riemannian metric; thus, we equip M with a Riemannian metric g.

If g has constant sectional curvature, then we are done:

Theorem. *Let M be a compact simply connected manifold of dimension n that admits a Riemannian metric of constant sectional curvature. Then M is homeomorphic to the n-dimensional sphere $\mathbb{S}^n$.*

As we pulled out g of an abstract hat, we cannot expect to be in the fortunate situation that g has constant sectional curvature. Following an idea of Hamilton [13],

- one can try to improve the potentially wild and "irregular" Riemannian metric g by applying a suitable flow to the Riemannian manifold (M, g), the so-called Ricci flow. The evolution of this flow is described by the following partial differential equation, involving the Ricci curvature:

$$\frac{\partial}{\partial_t} g_t = -2 \cdot \mathrm{Ric}^{g_t}$$

Roughly speaking, the Ricci flow has the tendency to deform Riemannian metrics into "rounder" Riemannian metrics. The difficulty is to control the singularities that might arise during this process and to ensure that this eventually leads to the desired situation of constant sectional curvature. This control was obtained by Perelman who thus completed Hamilton's program and proved the Poincaré conjecture (as well as Thurston's geometrisation conjecture).

For these achievements, Perelman has been awarded the Fields medal in 2006 (which he rejected) and a Clay Millenium Prize in 2010 (which he also rejected) [10].

Until today, no purely topological proof of the Poincaré conjecture (in dimension 3) is known!

While giving a full proof of the Poincaré conjecture is far beyond the scope of this course, we will meet several of the ingredients during this course (sectional curvature, Ricci curvature, ...); in particular, we will prove the fact that compact simply connected manifolds that admit a Riemannian metric of constant sectional curvature must be spheres – as well as the corresponding result for non-compact manifolds.

Overview of this course

We will start by recalling the basics of smooth manifolds; at this point, we will take the opportunity to also integrate the bundle language. We will then introduce the central notion of this course, Riemannian metrics, and spend some time on key examples.

After these preparations, we will use connections to introduce different notions of curvature of Riemannian manifolds and illustrate these in the case of the model spaces. The setup of connections also allows to treat geodesics and metric aspects of Riemannian manifolds efficiently.

Finally, we will study the global impact of curvature constraints in various settings, in particular, in relation with the fundamental group.

One of the main challenges in (differential) geometry is that geometrically intuitive definitions/arguments usually are hard to handle and that it is more efficient to replace them by analytic counterparts (which might be harder to visualise). During this course, I will try to also explain the geometric ideas underlying the analytic tools.

Study note (quick checks)**.** The quick checks in these lecture notes have feedback integrated into the pdf file. This feature is based on PDF layers (*not* on JavaScript) and is supported by many PDF viewers, such as Acrobat Reader, Evince, Foxit Reader, Okular, ... Let's test whether it works: Did you press the "No" button?

Yes No

Of course, you should only look at the hints or answers after you thought about the problem yourself; because you never know what you might unleash by prematurely clicking on a button.

Moreover, also other material will be provided in GRIPS and on the course homepage:

http://www.mathematik.uni-r.de/loeh/teaching/diffgeo_ws2021/

Study note (lecture notes). This course will be taught remotely, based on

- guided self-study of these lecture notes,
- interactive question sessions (video conferences and written forums),
- remote exercise sessions (with online submission and tutorials).

The details of this procedure are outlined in

http://www.mathematik.uni-r.de/loeh/teaching/diffgeo_ws2021/org.pdf

The lecture notes document the progress of the course, the topics covered in the course as well as some additional optional material.

This course will only treat the very beginning of this vast subject. It is therefore recommended to consult other sources (books and research articles!) for further information on this field.

References of the form "Satz I.6.4.11", "Satz II.2.4.33", "Satz III.2.2.25", "Satz IV.2.2.4", or "Corollary AT.1.3.25" point to the corresponding locations in the lecture notes for Linear Algebra I/II, Algebra, Commutative Algebra, Algebraic Topology in previuos semesters:

- http://www.mathematik.uni-r.de/loeh/teaching/linalg1_ws1617/lecture_notes.pdf
- http://www.mathematik.uni-r.de/loeh/teaching/linalg2_ss17/lecture_notes.pdf
- http://www.mathematik.uni-r.de/loeh/teaching/algebra_ws1718/lecture_notes.pdf
- http://www.mathematik.uni-r.de/loeh/teaching/calgebra_ss18/lecture_notes.pdf
- http://www.mathematik.uni-r.de/loeh/teaching/topologie1_ws1819/lecture_notes.pdf

Literature exercise. Where in the math library (including electronic resources) can you find books on differential/Riemannian geometry and neighbouring fields?

Literature exercise. In case the COVID-19 pandemic leaves you stranded at home with too much time for reading – here, you can find some suggestions:

http://www.mathematik.uni-r.de/loeh/teaching/diffgeo_ws2021/mustread.pdf

Convention. The set $\mathbb{N}$ of natural numbers contains 0. All rings are unital and associative (but very often *not* commutative). We write ${}_R\mathsf{Mod}$ for the category of left R-modules. Usually, we assume manifolds to be non-empty (but we might not always mention this explicitly).

1

Riemannian manifolds

A manifold is a space that is locally Euclidean. Interpreting "Euclidean" in the sense of spaces with a topology, smooth structure, or inner product leads to the notions of topological, smooth, and Riemannian manifolds, respectively.

We recall basic terminology on smooth manifolds and their tangent spaces. Moreover, we introduce the language of bundles; for us, the main examples will be bundles arising from the tangent bundle that allow to perform "smoothly parametrised" multilinear algebra.

In particular, we will use this language to formalise the notion of Riemannian metrics as "smoothly parametrised" inner products on the tangent spaces.

We illustrate these notions at the model spaces of Riemannian geometry: Euclidean spaces, spheres, and hyperbolic spaces. Finally, we sketch how Riemannian metrics are the basis of Riemannian *geometry*, a point of view that we will elaborate on for the rest of the course.

Overview of this chapter.

Running example. $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$

1.1 Smooth manifolds

A manifold is a space that is locally Euclidean. Interpreting “Euclidean” in the sense of spaces with a topology, smooth structure, or inner product leads to the notions of topological, smooth, and Riemannian manifolds, respectively.

We first recall the topological and the smooth case, including tangent spaces. In the next section, we will introduce the language of smooth vector bundles and construct the tangent bundle of a smooth manifold.

1.1.1 Topological manifolds

A topological manifold is a topological space that is locally homeomorphic to Euclidean space (Figure 1.1). In order to avoid pathologies, we also add two other topological conditions:

Definition 1.1.1 (topological manifold). Let $n \in \mathbb{N}$. A *topological manifold of dimension* n is a topological space M with the following properties:

- For each $x \in M$, there exists an open neighbourhood of x in M that is homeomorphic to an open subset of $\mathbb{R}^n$.
- The topological space M is Hausdorff and second countable.

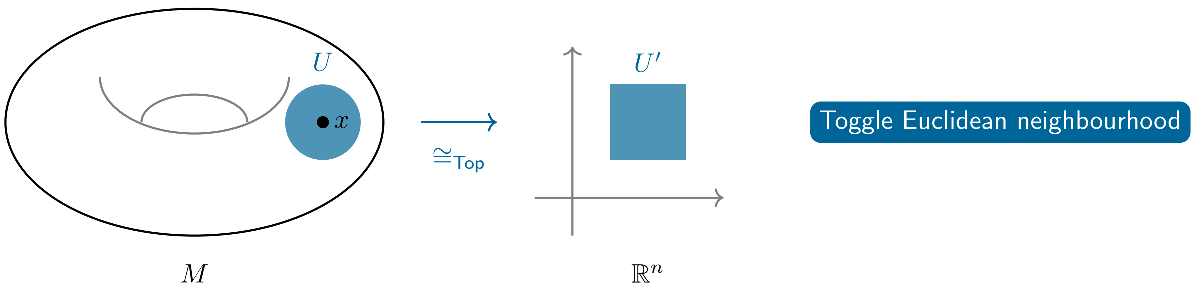


Figure 1.1.: A topological manifold, schematically

Remark 1.1.2 (dimension). If M is a non-empty topological manifold, then the dimension of M is well-defined (Corollary AT.3.2.10, Corollary AT.4.4.2).

Quick check 1.1.3 (on the local condition). Let $n \in \mathbb{N}$ and let M be a Hausdorff and second countable topological space. Is then M being a topological manifold equivalent to the following conditions?

1. For each $x \in M$, there exists an open neighbourhood of x in M that is homeomorphic to $\mathbb{R}^n$.

   Yes No

2. For each $x \in M$ and each open neighbourhood $U \subset M$ of x in M, there exists an open neighbourhood $V \subset U$ of x in M that is homeomorphic to an open subset of $\mathbb{R}^n$.

   Yes No

Example 1.1.4 (topological manifolds).

- For $n \in \mathbb{N}$, the vector space $\mathbb{R}^n$ with the standard topology is a topological manifold of dimension n.
- Open subspaces of topological manifolds are topological manifolds (with respect to the subspace topology).
- Products of finitely many topological manifolds are topological manifolds (with respect to the product topology).
- The space $(\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}) \subset \mathbb{R}^2$ with the subspace topology of $\mathbb{R}^2$ is *not* a topological manifold (Exercise).

1.1.2 Smooth manifolds

A smooth manifold is a topological space that is locally modelled on Euclidean space with the standard notion of smoothness (Figure 1.2):

- Each point is contained in the domain of a chart.
- The charts are smoothly compatible on the Euclidean side.

Depending on the choice of regularity on the Euclidean side, one can consider different levels of differentiability of manifolds: C^1, C^2, ... In order to keep the setup and notation simple, we will stick to the smooth (i.e., C^∞) case.

Definition 1.1.5 (chart). Let M be a topological manifold of dimension n.

- A *chart* of M is a homeomorphism $U \longrightarrow U'$ between open subsets $U \subset M$ and $U' \subset \mathbb{R}^n$. We call it a chart *around* $x \in M$ if $x \in U$.

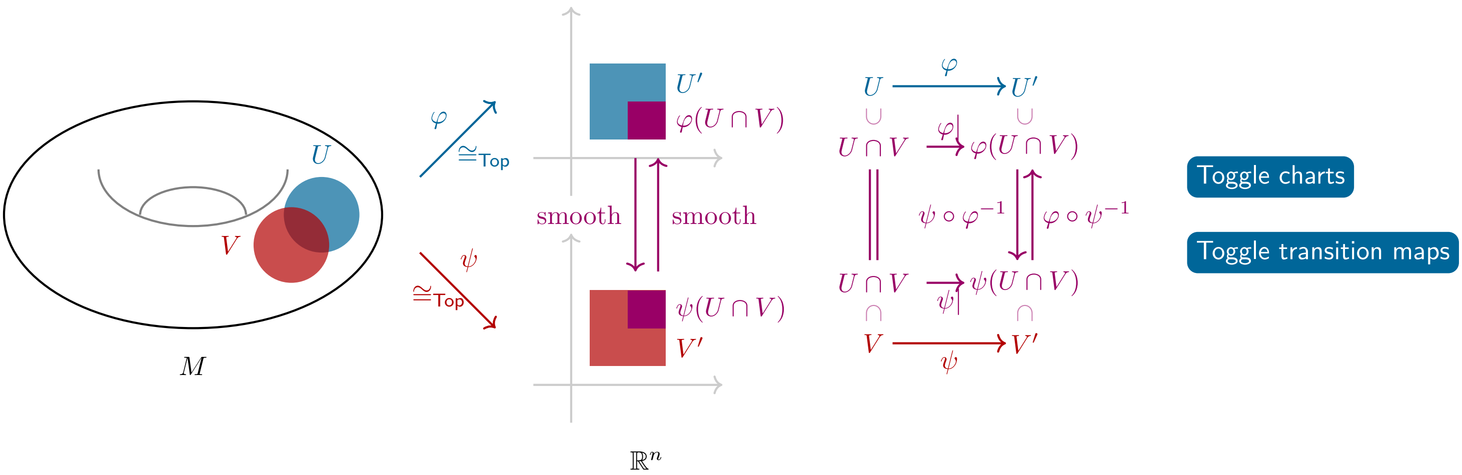


Figure 1.2.: Compatible charts, schematically

- Charts $\varphi\colon U \longrightarrow U'$ and $\psi\colon V \longrightarrow V'$ of M are *smoothly compatible* if the *transition maps*

$$\begin{aligned} \psi \circ \varphi^{-1}|_{\varphi(U \cap V)}&\colon \varphi(U \cap V) \longrightarrow \psi(U \cap V) \\ \varphi \circ \psi^{-1}|_{\psi(U \cap V)}&\colon \psi(U \cap V) \longrightarrow \varphi(U \cap V) \end{aligned}$$

  are smooth maps (in the sense of classical multi-variable analysis).

Definition 1.1.6 (atlas)**.** Let M be a topological manifold.

- An *atlas* for M is a set A of charts on M such that the domains of the charts in A cover all of M.

- An atlas A for M is *smooth* if any two charts in A are smoothly compatible.

- A smooth atlas A for M is *maximal* if the following holds: If A' is a smooth atlas on M that contains A, then $A = A'$. A maximal smooth atlas on M is also called a *smooth structure* on M.

Definition 1.1.7 (smooth manifold)**.** A *smooth manifold* is a pair (M, A), consisting of a topological manifold M and a smooth structure A on M.

Usually, we will not mention the smooth structure A explicitly in the notation and just say that "M is a smooth manifold"; charts in the smooth structure will just be called *smooth charts* of M.

Every smooth atlas is contained in a unique maximal smooth atlas (check!). Therefore, in order to specify a smooth structure on a topological manifold it suffices to specify a smooth atlas.

Example 1.1.8 (vector spaces as smooth manifolds)**.** Let $n \in \mathbb{N}$. Then the one-element set $\{\mathrm{id}_{\mathbb{R}^n} \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n\}$ is a smooth atlas for $\mathbb{R}^n$. We will always equip $\mathbb{R}^n$ with this smooth structure.

Moreover, every finite-dimensional vector space carries a canonical smooth structure: If V is an $\mathbb{R}$-vector space with $\dim V = n$, then every $\mathbb{R}$-linear isomorphism $\varphi \colon V \longrightarrow \mathbb{R}^n$ induces a topology on V, which turns V into a topological manifold of dimension n and φ into a chart for V. Moreover, any two such charts of V are smoothly compatible (Why?

). Therefore, $\{\varphi\}$ is a smooth atlas on V and the smooth structure on V generated by this atlas is independent of the choice of the linear isomorphism φ.

Quick check 1.1.9 (smooth structure on $\mathbb{R}$)**.**

1. Does the topological manifold $\mathbb{R}$ admit a smooth structure that consists of finitely many charts?

   Yes No

2. Does the standard smooth structure on $\mathbb{R}$ contain the map $\mathbb{R} \longrightarrow \mathbb{R}$ given by $x \longmapsto x^3$?

   Yes No

Example 1.1.10 (sub-manifolds)**.**

- Let $n \in \mathbb{N}$ and let $U \subset \mathbb{R}^n$ be open. Then U is a topological manifold (with respect to the subspace topology) and $\{\mathrm{id}_U \colon U \longrightarrow U\}$ is a smooth atlas on U. This turns U into a smooth manifold of dimension n.
- A convenient way to construct concrete examples of smooth manifolds is as submanifolds of Euclidean space (Chapter 1.1.5). In particular, this allows a very quick introduction of the spheres as smooth manifolds (Example 1.1.33).

Example 1.1.11 (products)**.** Let M and N be smooth manifolds. Then, taking cartesian products of smooth charts on M and N, respectively, defines a smooth atlas on $M \times N$ (check!).

Remark 1.1.12 (local coordinates, parametrisation)**.** Let M be a topological manifold of dimension n and let $\varphi \colon U \longrightarrow U'$ be a smooth chart on M. The

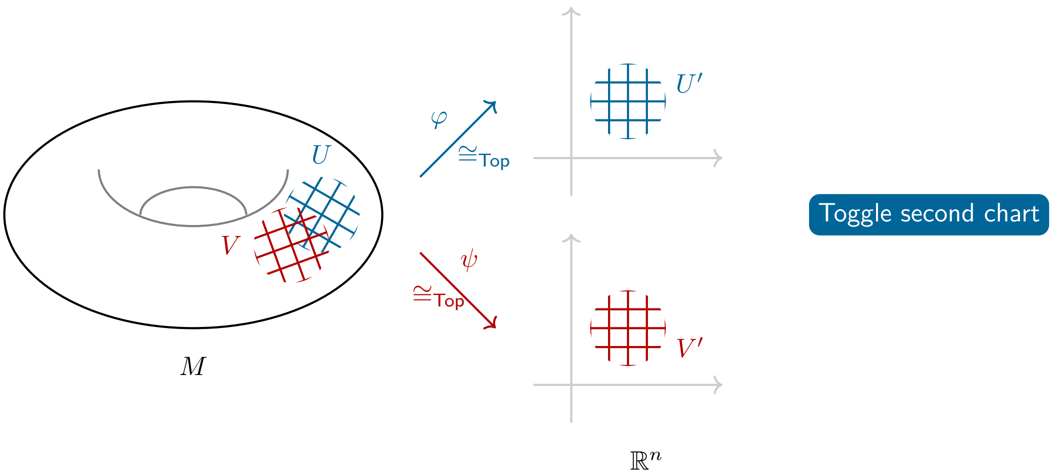


Figure 1.3.: Local coordinates, schematically

n component functions $U \longrightarrow \mathbb{R}$ of φ are called *local coordinates* on U for M with respect to the chart φ (Figure 1.3). If the chosen chart is clear from the context, it is common to write

$$(x^1, \dots, x^n) := \varphi(x) \in \mathbb{R}^n$$

for the φ-coordinates of points $x \in U$. The choice of upper indices is convenient in view of the *Einstein summation convention* (Remark 1.1.13). Sometimes, one goes one step further and denotes the coordinate *functions* of φ by $x^1, \dots, x^n \colon U \longrightarrow \mathbb{R}$.

The inverse homeomorphism $U' \longrightarrow U$ of φ is usually termed a *parametrisation* of U in M.

Remark 1.1.13 (Einstein summation convention)**.** Let $n \in \mathbb{N}$, let V be an $\mathbb{R}$-vector space, and let $a^1, \dots, a^n \in \mathbb{R}$, $v_1, \dots, v_n \in V$. Then we abbreviate

$$a^j \cdot v_j := \sum_{j=1}^{n} a^j \cdot v_j.$$

I.e., free variables on the left-hand side that occur once in lower and once in upper position will trigger a summation process of the products over an appropriate range. This can be extended in virtuous ways: For instance, one can apply this to multiple indices at once; for denominators, the meaning of upper and lower indices is flipped.

This notation is extremely concise and convenient in differential geometry. But there is also the risk of misunderstandings. Therefore, when in doubt: Use the "normal" sum notation! In these lecture notes, when resorting to Einstein summation, I will use the following non-standard additional convention to indicate the use of Einstein summation convention:

$$\underbracket{a^j \cdot v_j} := \sum_{j=1}^{n} a^j \cdot v_j.$$

Moreover, whenever possible, the use of coordinates should be avoided and coordinate-free or intrinsic descriptions should be preferred.

Quick check 1.1.14. Let $n \in \mathbb{N}$. We consider the real numbers $a^j := j$, $v_j := 1$, $w_j := j$ for all $j \in \{1, \dots, n\}$. Expand the following terms:

1. $\underbracket{a^j \cdot v_j}$ Check!
2. $\underbracket{a^j \cdot w_j}$ Check!
3. $\underbracket{a^k \cdot v_k}$ Check!
4. $\underbracket{a^1 \cdot v_j}$ Check!

1.1.3 The category of smooth manifolds

As always, objects need to be related by suitable, structure preserving, morphisms. In the case of smooth manifolds, this leads to the notion of smooth maps and whence to the category of smooth manifolds. Smooth maps are maps whose local descriptions via charts are smooth maps between open subsets of Euclidean spaces (Figure 1.4).

Definition 1.1.15 (smooth map)**.** A map $f \colon M \longrightarrow N$ between smooth manifolds is *smooth* if the following holds: For each $x \in M$, there exists a smooth chart $\varphi \colon U \longrightarrow U'$ of M around x and a smooth chart $\psi \colon V \longrightarrow V'$ of N around $f(x)$ with $f(U) \subset V$ such that the map

$$\psi \circ f \circ \varphi^{-1} \colon U' \longrightarrow V'$$

is smooth (in the sense of classical multi-variable analysis).

We write $C^\infty(M, N)$ for the set of all smooth maps $M \longrightarrow N$. Moreover, we write $C^\infty(M) := C^\infty(M, \mathbb{R})$ for the set of smooth maps $M \longrightarrow \mathbb{R}$ (which is an $\mathbb{R}$-algebra with respect to pointwise operations; check!).

If $f \colon M \longrightarrow N$ is a smooth map and $\varphi \colon U \longrightarrow U'$ and $\psi \colon V \longrightarrow V'$ are smooth charts of M and N, respectively, with $f(U) \subset V$, then $\psi \circ f \circ \varphi^{-1} \colon U' \longrightarrow V'$ is smooth (check!); i.e., the notion of smoothness of a map does not depend on the chosen charts (provided they belong to the same smooth structure).

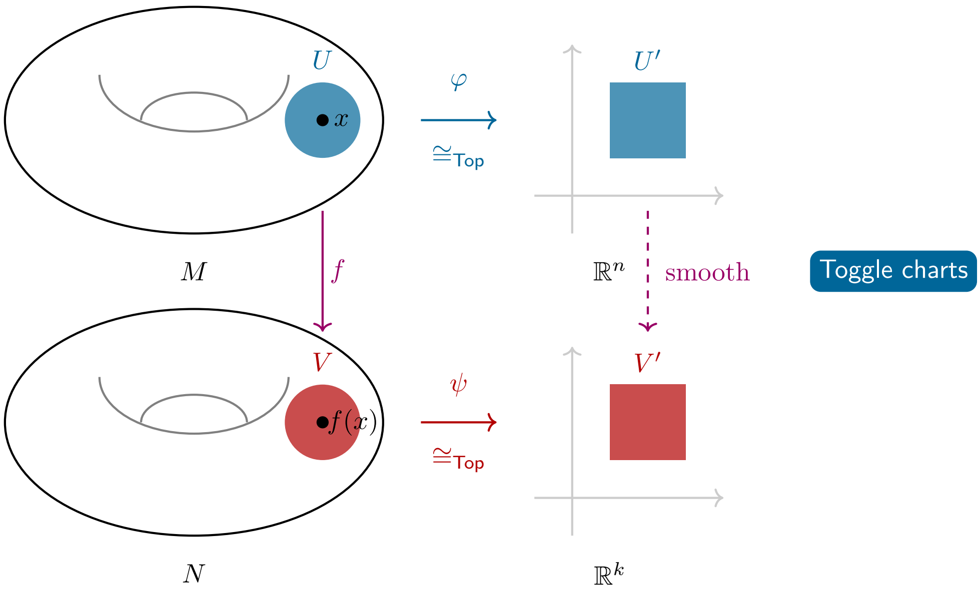


Figure 1.4.: Smooth maps, schematically

Example 1.1.16 (smooth maps)**.** Let M be a smooth manifold.

- The identity map $\mathrm{id}_M\colon M \longrightarrow M$ is a smooth map (check!); we implicitly used the *same* smooth structure on M in the domain and target.
- If $f\colon M \longrightarrow N$ and $g\colon N \longrightarrow P$ are smooth maps between smooth manifolds, then $g \circ f\colon M \longrightarrow P$ is also a smooth map (check!).
- If $I \subset \mathbb{R}$ is an interval, then smooth maps $I \longrightarrow M$ are called *smooth curves on M*.
- If N is a smooth manifold and $y \in N$, then the constant map that maps M to y is a smooth map $M \longrightarrow N$ (check!).

Remark 1.1.17 (the category of smooth manifolds)**.** The category Mfd consists of the following data:

- Objects: The class of objects is the class of all smooth manifolds.
- Morphisms: If M and N are smooth manifolds, then we set
$$\mathrm{Mor}_{\mathsf{Mfd}}(M, N) := C^\infty(M, N).$$
- Compositions: The compositions are given by the usual compositions of smooth maps.

Definition 1.1.18 (diffeomorphism)**.** A *diffeomorphism* is an isomorphism in the category Mfd of smooth manifolds. More explicitly: Let M and N be smooth manifolds. A *diffeomorphism* $M \longrightarrow N$ is a smooth map $f \colon M \longrightarrow N$ for which there exists a smooth map $g \colon N \longrightarrow M$ with

$$g \circ f = \mathrm{id}_M \quad \text{and} \quad f \circ g = \mathrm{id}_N \,.$$

Example 1.1.19 (charts as diffeomorphisms)**.** Let M be a smooth manifold. If $\varphi \colon U \longrightarrow U'$ is a smooth chart of M, then φ is a diffeomorphism, where U carries the smooth structure inherited from M and U' carries the smooth structure inherited from $\mathbb{R}^{\dim M}$ (check!).

Quick check 1.1.20. Let M be smooth manifold. Why didn't we define charts from the very beginning as diffeomorphisms between open subsets of M and open subsets of a Euclidean space? Why?

1.1.4 Tangent spaces

The defining property of differentiable maps between Euclidean spaces is local approximability by linear maps; in the differentiable case, these local approximations are uniquely determined and form the differential/derivative. As this is a local construction, it can also be carried out for smooth manifolds:

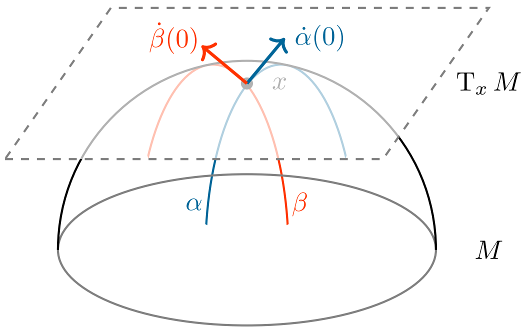
- We first need suitable vector spaces: The tangent spaces. We will define these geometrically via the "velocity" of curves, using charts and the $\mathbb{R}$-linear structure on Euclidean spaces as well as the notion of derivatives for maps between Euclidean spaces (Figure 1.5).
- We then consider the effect of smooth maps on these tangent spaces. Geometrically, this comes from the local action on curves.

Definition 1.1.21 (tangent vector)**.** Let M be a smooth manifold, let $x \in M$, and let $I_0(\mathbb{R})$ denote the set of all open intervals in $\mathbb{R}$ that contain 0.

- We write $C(M;x)$ for the set of all smooth curves $\alpha \colon I \longrightarrow M$ with $I \in I_0(\mathbb{R})$ and $\alpha(0) = x$.
- Smooth curves $(\alpha \colon I \longrightarrow M), (\beta \colon J \longrightarrow M) \in C(M;x)$ are *equivalent* if there exists a smooth chart $\varphi \colon U \longrightarrow U'$ around x such that

  $$(\varphi \circ \alpha)'(0) = (\varphi \circ \beta)'(0).$$

  The equivalence classes are called *tangent vectors at x*.
- The tangent vector represented by a smooth curve $\alpha \in C(M;X)$ is denoted by $\dot\alpha(0)$.



Toggle tangent space

Figure 1.5.: The tangent space, schematically

Proposition 1.1.22 (tangent space). *Let M be a smooth manifold of dimension n and let $x \in M$.*

1. *The notion of equivalence of Definition 1.1.21 is an equivalence relation on $C(M;x)$. We write*

$$\mathrm{T}_x\,M := C(M;x)/{\sim}.$$

2. *Let $\varphi\colon U \longrightarrow U'$ be a smooth chart for M around x. Then the maps*

$$\begin{aligned} \mathrm{T}_x\,M &\longrightarrow \mathbb{R}^n \\ [\alpha] &\longmapsto (\varphi \circ \alpha)'(0), \\ \mathbb{R}^n &\longrightarrow \mathrm{T}_x\,M \\ v &\longmapsto \bigl[(t \mapsto \varphi^{-1}(t \cdot v + \varphi(x)))\colon (-\varepsilon,\varepsilon) \to M\bigr], \\ &\qquad \textit{where } \varepsilon \in \mathbb{R}_{>0} \textit{ is small enough} \end{aligned}$$

   are well-defined and mutually inverse.

3. *The set $\mathrm{T}_x\,M$ carries a canonical $\mathbb{R}$-vector space structure (of dimension n): The $\mathbb{R}$-vector space structure on $\mathrm{T}_x\,M$ induced by the $\mathbb{R}$-vector structure on $\mathbb{R}^n$ with respect to the maps from 2. is independent of the chosen chart around x.*

Proof. These are straightforward (but lengthy) computations (check!) [1, Chapter 2.3]. □

Quick check 1.1.23. How can one express the addition and scalar multiplication of tangent vectors explicitly in terms of curves and charts?

Hint

Example 1.1.24 (tangent spaces of vector spaces)**.** Let V be a finite-dimensional $\mathbb{R}$-vector space. We can view V as a smooth manifold (Example 1.1.8). For each $x \in V$, the map

$$\begin{aligned} V &\longrightarrow \mathrm{T}_x V \\ v &\longmapsto \dot{\alpha}_v(0), \text{ where } \alpha_v \colon t \mapsto t \cdot v + x \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism (check!). Under this isomorphism, classical multivariable analysis can be translated into the setting of smooth manifolds and tangent spaces.

Remark 1.1.25 (tangent spaces via derivations)**.** The definition of the tangent space given in Proposition 1.1.22 is geometrically intuitive. However, it is sometimes more convenient to use the following description (which is more amenable to other formal methods):

Let M be a smooth manifold and let $x \in M$. A *derivation at* x is an $\mathbb{R}$-linear map $D \colon C^\infty(M) \longrightarrow \mathbb{R}$ with the following property:

$$\forall_{f,g \in C^\infty(M)} \quad D(f \cdot g) = D(f) \cdot g(x) + f(x) \cdot D(g)$$

For example, if $v \in \mathrm{T}_x M$ is represented by the curve $\alpha \in C(M;x)$, then

$$\begin{aligned} \partial_v \colon C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto (f \circ \alpha)'(0) \end{aligned}$$

is independent of the representative α and a derivation at x (check!).

We write $D(M;x)$ for the set of all derivations at x. Then, $D(M;x)$ is an $\mathbb{R}$-vector space (with respect to pointwise operations) and

$$\begin{aligned} \mathrm{T}_x M &\longrightarrow D(M;x) \\ v &\longmapsto \partial_v \end{aligned}$$

is an isomorphism of $\mathbb{R}$-vector spaces [1, Satz 2.35].

Definition 1.1.26 (differential)**.** Let $f \colon M \longrightarrow N$ be a smooth map between smooth manifolds and let $x \in M$. Then the *differential of* f *at* x is the $\mathbb{R}$-linear map (check!) given by

$$\begin{aligned} \mathrm{d}_x f \colon \ \mathrm{T}_x M &\longrightarrow \mathrm{T}_{f(x)}(N) \\ \dot{\alpha}(0) &\longmapsto \dot{\beta}(0), \text{ where } \beta := f \circ \alpha. \end{aligned}$$

Example 1.1.27 (differentials)**.** Let M be a smooth manifold and let $x \in M$.

- Then $\mathrm{d}_x \mathrm{id}_M =$? (check!).
- If $f \colon M \longrightarrow N$ and $g \colon N \longrightarrow P$ are smooth maps, then (check!)

$$\mathrm{d}_x(g \circ f) = \text{?} \qquad .$$

- If V, W are finite-dimensional vector spaces and $f\colon V \longrightarrow W$ is an $\mathbb{R}$-linear map, then f is smooth (check!) and
$$\forall_{x \in V} \quad \mathrm{d}_x f = \boxed{?}$$
(check!); here, we use the identification $\mathrm{T}_x V \cong_{\mathbb{R}} V$ of Example 1.1.24.

Remark 1.1.28 (local coordinates)**.** Let M be a smooth manifold of dimension n, let $x \in M$, and let $\varphi\colon U \longrightarrow U'$ be a chart around x. Then the local coordinates on U induced by the chart φ (Remark 1.1.12) lead to a corresponding basis
$$\bigl(\partial_j := \mathrm{d}_x(\varphi^{-1})(e_j)\bigr)_{j \in \{1,\dots,n\}}$$
of $\mathrm{T}_x M$ (check!). In concrete examples, it can be helpful to express differentials of smooth maps in terms of such coordinates.

1.1.5 Submanifolds

For the sake of concreteness, it can be convenient to specify manifolds as submanifolds of Euclidean spaces.

- Submanifolds of Euclidean spaces are subspaces that locally can be described in terms of submanifold charts (also called slice charts; Figure 1.6).
- Equivalently, smooth submanifolds can be obtained locally as sets of solutions of suitable equations (by the regular value theorem; Theorem 1.1.32) or as local parametrisations.
- Moreover, smooth submanifolds can be characterised in terms of immersions (Theorem 1.1.36).

Definition 1.1.29 (submanifold chart, submanifold)**.** Let $N, n \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a subset.

- Let $x \in M$. An *n-dimensional smooth submanifold chart for M around x* is a diffeomorphism $\varphi\colon U \longrightarrow U'$ between open subsets $U, U' \subset \mathbb{R}^N$ with $x \in U$ and
$$\varphi(U \cap M) = U' \cap (\mathbb{R}^n \times \{0\}).$$
- We call M an n-dimensional *smooth submanifold of* $\mathbb{R}^N$, if every point in M admits an n-dimensional smooth submanifold chart around it.

Remark 1.1.30 (smooth structures from submanifold charts)**.** Let $N \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a smooth submanifold of dimension n. Then

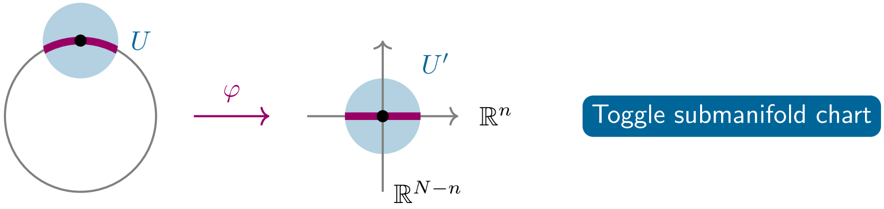


Figure 1.6.: Smooth submanifolds, schematically

$$\{\varphi|_{U\cap M} \colon U \cap M \to \varphi(U\cap M) \mid \varphi \colon U \to U' \text{ smooth submanifold chart on } M\}$$

is a smooth atlas on M (check!). We will always equip submanifolds with this induced smooth structure.

Remark 1.1.31 (tangent space of a submanifold)**.** The tangent space of a submanifold admits the following explicit description: Let $N \in \mathbb{N}$, let $M \subset \mathbb{R}^N$ be a smooth submanifold, and let $x \in M$. Then

$$\begin{aligned} \mathrm{T}_x M &\longrightarrow \bigl\{\gamma'(0) \bigm| \gamma \in C(\mathbb{R}^N; x),\ \operatorname{im}\gamma \subset M\bigr\} \\ \dot\alpha(0) &\longmapsto \alpha'(0) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism (check!).

Theorem 1.1.32 (regular value theorem [1, Satz 1.4])**.** *Let $N, n \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a subset. Then the following are equivalent:*

1. *The subset $M \subset \mathbb{R}^N$ is a smooth n-dimensional submanifold.*
2. *For every $x \in M$, there exists an open neighbourhood $U \subset \mathbb{R}^N$ of x and a smooth map $f \in C^\infty(U, \mathbb{R}^{N-n})$ with the following properties:*
   - 0 *is a* regular value of f, *i.e., for all $z \in f^{-1}(0)$, we have* $\operatorname{rk} f'(z) = N - n$, *and*
   - $f^{-1}(0) = U \cap M$.
3. *There exist local parametrisations for M, i.e.: For every $x \in M$, there exists a permutation $\pi \in \Sigma_N$, open subsets $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^{N-n}$ such that $V \times W$ is an open neighbourhood of x, and a map $f \in C^\infty(V, W)$ with*
   $$(V \times W) \cap L_\pi(M) = \textit{graph of } f,$$
   where $L_\pi \colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is the coordinate permutation induced by π.

Example 1.1.33 (spheres as submanifolds)**.** Let $n \in \mathbb{N}$. Then 0 is a regular value of the map

$$\begin{aligned}\mathbb{R}^{n+1} &\longrightarrow \mathbb{R}\\ x &\longmapsto \|x\|_2{}^2 - 1\end{aligned}$$

(check!). Therefore, the *n-sphere*

$$\mathbb{S}^n := \bigl\{x \in \mathbb{R}^{n+1} \bigm| \|x\|_2 = 1\bigr\}$$

is a smooth submanifold of $\mathbb{R}^{n+1}$. We will always equip $\mathbb{S}^n$ with the corresponding smooth structure.

Study note. It is instructive to also give local parametrisations for spheres and to describe them as "abstract" smooth manifolds without referring to the regular value theorem (e.g., using the stereographic projections for the North and the South pole).

Definition 1.1.34 (immersion, submersion)**.** Let $f\colon M \longrightarrow N$ be a smooth map between smooth manifolds.

- The map f is an *immersion* if $\mathrm{d}_x f\colon \mathrm{T}_x M \longrightarrow \mathrm{T}_{f(x)} N$ is injective for every $x \in M$
- and f is a *submersion* if $\mathrm{d}_x f\colon \mathrm{T}_x M \longrightarrow \mathrm{T}_{f(x)} N$ is surjective for every $x \in M$.

Quick check 1.1.35 (immersions/submersions)**.**

1. Is every immersion injective?

   Yes No

2. Is every submersion surjective?

   Yes No

Theorem 1.1.36 (submanifolds via immersions [1, (generalisation of) Satz 1.11])**.** *Let $N \in \mathbb{N}$, let W be a smooth manifold and let $f\colon W \longrightarrow \mathbb{R}^N$ be an injective smooth immersion. Then the following are equivalent:*

1. *The set $f(W)$ is a smooth submanifold of $\mathbb{R}^N$.*
2. *The map $f\colon W \longrightarrow f(W)$ is a homeomorphism (with respect to the subspace topology on $f(W)$).*

Outlook 1.1.37 (exotic smooth structures)**.** It is known that all topological manifolds of dimension ≤ 3 admit a smooth structure; moreover, these smooth structures are unique up to diffeomorphism [25]. However, in higher dimensions, the picture is more subtle. By a famous result of Milnor [22], the sphere $\mathbb{S}^7$ admits a smooth structure that is *not* diffeomorphic to the standard smooth structure on $\mathbb{S}^7$.

Outlook 1.1.38 (embedding theorems). Every smooth manifold M admits a smooth embedding into $\mathbb{R}^{2 \cdot \dim M}$ (Whitney embedding theorem). Thus, in principle, submanifolds of Euclidean spaces are as general as abstract smooth manifolds. However, as in every instance of the eternal battle between the concrete and the abstract, eventually both the concrete and the abstract descriptions will be necessary:

- In many cases, it is convenient to be able to define and work with smooth manifolds in a coordinate- and embedding-free way. Prototypical concrete examples of this type are the projective spaces $\mathbb{R}P^n$ (Example 1.1.10).
- In contrast, describing smooth manifolds as submanifolds of Euclidean space or as sets of solutions of equations allows, e.g., for the use of visualisation software.

Quick check 1.1.39 (visualisation). Visualise the submanifolds

$$\begin{gathered}
\bigl\{(x,y,z) \bigm| x^2+y^2+z^2=1\bigr\}\\
\bigl\{(x,y,z) \bigm| x^2+y^2-z^2=1\bigr\}\\
\bigl\{(x,y,z) \bigm| x^2+y^2+z^2=-1\bigr\}\\
\bigl\{(x,y,z) \bigm| x^2+y^2-z^2=-1\bigr\}
\end{gathered}$$

of $\mathbb{R}^3$ with appropriate visualisation software.

1.2 The tangent bundle

We now introduce the language of smooth vector bundles to properly formalise the notion of tangent bundles. Moreover, we will explain how to extend constructions from (multi)linear algebra to vector bundles. In particular, this will allow us to construct the various tensor bundles over smooth manifolds.

1.2.1 Smooth vector bundles

Smooth vector bundles are a parametrised version of linear algebra:

- A smooth vector bundle is an assembly of $\mathbb{R}$-vector spaces (the fibres) over a base manifold that combines into a smooth manifold (the so-called total space) (Figure 1.7). This situation is formalised as a smooth map from the total space to the base manifold satisfying a local triviality condition.
- A bundle map is a smooth map between the total spaces that is linear on the fibres. This leads to the category of smooth vector bundles.

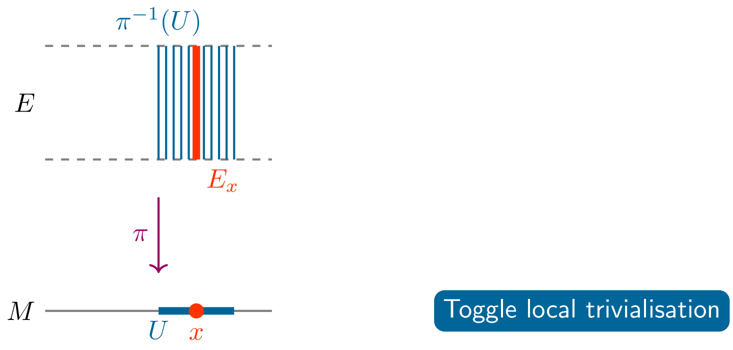


Figure 1.7.: Vector bundles, schematically

Definition 1.2.1 (smooth vector bundle)**.** Let M be a smooth manifold and let $k \in \mathbb{N}$. A *smooth vector bundle over M of rank k* is a smooth map $\pi\colon E \longrightarrow M$ between smooth manifolds with the following data/properties:

- For each $x \in M$, the *fibre* $E_x := \pi^{-1}(\{x\}) \subset E$ is endowed with the structure of an $\mathbb{R}$-vector space.
- For each $x \in M$, there exists a *local trivialisation of π around x*, i.e., there exists an open neighbourhood U of $x \in M$ in M and a diffeomorphism $\varphi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ such that:
  - we have $p \circ \varphi = \pi|_U$, where $p\colon U \times \mathbb{R}^k \longrightarrow U$ is the projection onto the first factor
$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi|"'] & U \times \mathbb{R}^k \arrow[d, "p"] \\ U \arrow[r, equal] & U \end{tikzcd}$$
  - and for each $y \in U$, the restriction $\varphi|_{E_y}\colon E_y \longrightarrow \{y\} \times \mathbb{R}^k$ is an $\mathbb{R}$-linear isomorphism, where $\{y\} \times \mathbb{R}^k$ carries the canonical $\mathbb{R}$-vector space structure of the second factor.

For brevity, we will usually just say "vector bundle" instead of "smooth vector bundle". Moreover, if the bundle projection π is clear from the context, one often abuses notation and speaks of the "vector bundle E over M".

One-dimensional vector bundles are also called *line bundles*.

Example 1.2.2 (trivial vector bundles)**.** Let M be a smooth manifold and let $k \in \mathbb{N}$. Then the projection

$$M \times \mathbb{R}^k \longrightarrow M$$

to the first factor is a smooth vector bundle over M of rank k (check!), "the" *trivial vector bundle over M of rank k.*

Definition 1.2.3 (bundle map)**.** Let $\pi\colon E \longrightarrow M$, $\pi'\colon E' \longrightarrow M'$ be smooth vector bundles. A *bundle map* $\pi \longrightarrow \pi'$ is a pair $(F\colon E \longrightarrow E', f\colon M \longrightarrow M')$ of smooth maps with the following properties:

- The maps are fibre-preserving, i.e., $\pi' \circ F = f \circ \pi$.
- The map F is fibrewise linear, i.e., for all $x \in M$, the restriction map $F|_{E_x}: E_x \longrightarrow E'_{f(x)}$ is $\mathbb{R}$-linear.

As f is determined by F, one often also speaks of F as a bundle map.

Remark 1.2.4 (the category of smooth vector bundles)**.** If $\pi\colon E \longrightarrow M$ is a smooth vector bundle, then $(\mathrm{id}_E, \mathrm{id}_M)$ is a bundle map; moreover, the fibrewise composition of bundle maps is a bundle map (check!). Therefore, smooth vector bundles and bundle maps form a category VectB of smooth vector bundles.

A smooth vector bundle over a smooth manifold M is called *trivial* if it is isomorphic (in the category VectB) to a trivial bundle in the sense of Example 1.2.2.

Definition 1.2.5 (section)**.** Let $\pi\colon E \longrightarrow M$ be a smooth vector bundle. A *section of* π is a smooth map $s\colon M \longrightarrow E$ with

$$\pi \circ s = \mathrm{id}_M,$$

i.e., s maps each point of M into the corresponding fibre of π. The set of all sections of π is denoted by $\Gamma(\pi)$ or alternatively by $\Gamma(M, E)$ or $\Gamma(E)$.

Remark 1.2.6 (the structure of the space of sections)**.** Let $\pi\colon E \longrightarrow M$ be a smooth vector bundle. Then, $\Gamma(\pi)$ is a $C^\infty(M)$-module:

- If $s_1, s_2\colon M \longrightarrow E$ are sections of π, then also

  $$\begin{aligned} s_1 + s_2\colon M &\longrightarrow E \\ x &\longmapsto s_1(x) + s_2(x) \end{aligned}$$

  is a section of π (check!). What does "+" mean on the right-hand side of the definition of $s_1 + s_2$?

  Hint!

- If $s\colon M \longrightarrow E$ is a section of π and $f \in C^\infty(M)$, then also

  $$\begin{aligned} f \cdot s\colon M &\longrightarrow E \\ x &\longmapsto f(x) \cdot s(x) \end{aligned}$$

  is a section of π (check!). What does "·" mean on the right-hand side of the definition of $f \cdot s$?

  Hint!

1.2.2 Constructing vector bundles

Like manifolds can be glued from charts, smooth vector bundles can be glued from local trivialisations; we only have to make sure that the fibres in spe fit together linearly. In order to figure out suitable compatibility conditions, we first look at the transition maps of existing vector bundles.

For an $\mathbb{R}$-vector space V, we denote the group of $\mathbb{R}$-automorphisms of V by $\mathrm{GL}(V)$. If V is finite-dimensional, then we view $\mathrm{GL}(V)$ as a smooth manifold How?

Proposition 1.2.7 (transition functions of vector bundles)**.** *Let $\pi\colon E \longrightarrow M$ be a smooth vector bundle of rank k, and let $\varphi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$, $\psi\colon \pi^{-1}(V) \longrightarrow V \times \mathbb{R}^k$ be smooth local trivialisations of π over open subsets $U, V \subset M$ with $U \cap V \neq \emptyset$.*

Then there exists a smooth map $\tau\colon U \cap V \longrightarrow \mathrm{GL}(\mathbb{R}^k)$ satisfying

$$\forall_{x \in U \cap V} \quad \forall_{v \in \mathbb{R}^k} \quad \varphi \circ \psi^{-1}(x, v) = \bigl(x, \tau(x)(v)\bigr).$$

We call τ the transition function between φ and ψ.

Proof. We only need to unpack the notion of local trivialisations: Let

$$\sigma := (\text{projection onto the } \mathbb{R}^k\text{-factor}) \circ \varphi \circ \psi^{-1}\colon (U \cap V) \times \mathbb{R}^k \longrightarrow \mathbb{R}^k.$$

Because φ and ψ are local trivialisations of the smooth vector bundle π, the map σ is smooth and for each $x \in U \cap V$, the map $\sigma(x, \,\cdot\,)\colon \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is an $\mathbb{R}$-linear isomorphism. Therefore,

$$\begin{aligned} \tau\colon U \cap V &\longrightarrow \mathrm{GL}(\mathbb{R}^k) \\ x &\longmapsto \sigma(x, \,\cdot\,) \end{aligned}$$

is a well-defined map; moreover, smoothness of σ leads to smoothness of τ (Exercise). By construction, τ satisfies the transition function equation for φ and ψ. □

Proposition 1.2.8 (construction lemma for vector bundles)**.** *Let M be a smooth manifold, let $k \in \mathbb{N}$,*

- *let $(E_x)_{x \in M}$ be a family of $\mathbb{R}$-vector spaces of dimension k,*

let $E := \coprod_{x \in M} E_x$ be the corresponding disjoint union, and let $\pi\colon E \longrightarrow M$ be the index projection map. Furthermore,

- *let $(U_i)_{i \in I}$ be an open cover of M,*
- *let $(\varphi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k)_{i \in I}$ be given, and*

- *let* $(\tau_{i,j} \colon U_i \cap U_j \to \mathrm{GL}(\mathbb{R}^k))_{i,j \in I}$ *be a family of smooth maps*

with the following properties:

- *For each* $i \in I$*, the map* $\varphi_i \colon \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$ *is bijective,*
- *for each* $i \in I$ *and each* $x \in U_i$*, we have* $\varphi_i(E_x) \subset \{x\} \times \mathbb{R}^k$ *and* $\varphi_i|_{E_x} \colon E_x \longrightarrow \{x\} \times \mathbb{R}^k$ *is an* $\mathbb{R}$*-linear isomorphism, and*
- *for all* $i, j \in I$*, all* $x \in U_i \cap U_j$ *and all* $v \in E_x$*, we have*
$$\varphi_i \circ \varphi_j^{-1}(x, v) = \big(x, \tau_{i,j}(x)(v)\big).$$

Then, there exists a unique smooth structure on E *with the property that* $\pi \colon E \longrightarrow M$ *is a smooth vector bundle of rank* k *such that* $(\varphi_i)_{i \in I}$ *are local trivialisations of* π *and such that the* $(\tau_{i,j})_{i,j \in I}$ *are the corresponding transition functions.*

Proof. We first indicate how to construct a smooth structure on E and then we will explain why it has the desired properties.

Construction of a smooth structure on E. The construction uses the local structure of E as "neighbourhoods in M times $\mathbb{R}^k$". Let $x \in M$. Then there exists an $i \in I$ with $x \in U_i$. Moreover, we choose a smooth chart $\psi_x \colon V_x \longrightarrow V'_x$ of M around x with $V_x \subset U_i$ and consider the map

$$\widetilde{\psi}_x := (\psi_x \times \mathrm{id}_{\mathbb{R}^k}) \circ \varphi_i|_{\pi^{-1}(V_x)} \colon \pi^{-1}(V_x) \longrightarrow V'_x \times \mathbb{R}^k.$$

Then $\{\widetilde{\psi}_x \mid x \in M\}$ induces a topology on E and forms a smooth atlas for E (check! [17, proof of Lemma 5.5]). We equip E with the smooth structure generated by this smooth atlas.

Smoothness properties of the maps. With respect to this smooth structure, we have:

- For $i \in I$, the map $\varphi_i \colon \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$ is a diffeomorphisms (because with respect to the smooth charts used in the construction of the smooth structure on E, the map φ_i is locally represented by identity maps).
- The map π is smooth (again, locally with respect to these charts the map is the projection of a cartesian product onto a factor).

By construction, $\pi \colon E \longrightarrow M$ is then a smooth vector bundle with local trivialisations $(\varphi_i)_{i \in I}$ and with transition functions $(\tau_{i,j})_{i,j \in I}$ (check!).

Uniqueness property. Let A and A' be smooth structures on E with the desired properties. We show that $A = A'$: The identity map id_E locally can be described as "$\varphi_i \circ \varphi_i^{-1}$". Therefore, $\mathrm{id}_E \colon (E, A) \longrightarrow (E, A')$ is a bijective map that is a local diffeomorphism. Hence, it must be a diffeomorphism (Exercise). Thus, $A = A'$, which proves uniqueness. ☐

Corollary 1.2.9 (vector bundles from cocycles). *Let M be a smooth manifold, let $(U_i)_{i\in I}$ be an open cover of M, let V be a finite-dimensional $\mathbb{R}$-vector space, and let $(\tau_{i,j}\colon U_i \cap U_j \to \mathrm{GL}(V))_{i,j\in I}$ be a family of smooth maps satisfying the* cocycle condition*:*

$$\forall_{i,j,h\in I} \quad \forall_{x\in U_i\cap U_j\cap U_h} \quad \tau_{i,j}(x)\circ\tau_{j,h}(x) = \tau_{i,h}(x)$$

Then there exists a smooth vector bundle over M of rank $\dim_{\mathbb{R}} V$ with local trivialisations over the $(U_i)_{i\in I}$ and whose transition functions are the given functions $(\tau_{i,j})_{i,j\in I}$. Moreover, this bundle is unique up to bundle isomorphism.

Remark 1.2.10 (consequences of the cocycle condition). Before going into the proof of Corollary 1.2.9, we first have a closer look at the cocycle condition. In the situation of Corollary 1.2.9, the cocycle condition has the following consequences:

- For all $i \in I$ and all $x \in U_i$, we have $\tau_{i,i}(x) = \mathrm{id}_V$ because

$$\begin{aligned} \tau_{i,i}(x) &= (\tau_{i,i}(x))^{-1}\circ\tau_{i,i}(x)\circ\tau_{i,i}(x) && \text{(invertibility)} \\ &= (\tau_{i,i}(x))^{-1}\circ\tau_{i,i}(x) && (\text{Why?}) \\ &= \mathrm{id}_V\,. \end{aligned}$$

- For all $i,j \in I$ and all $x \in U_i\cap U_j$, we have $\tau_{i,j}(x) = (\tau_{j,i}(x))^{-1}$ because

$$\begin{aligned} \tau_{i,j}(x)\circ\tau_{j,i}(x) &= \tau_{i,i}(x) && \text{(by the cocycle condition)} \\ &= \mathrm{id}_V; && \text{(by the first part)} \end{aligned}$$

  by symmetry, the other composition also results in id_V.

Proof of Corollary 1.2.9. We will use the vector bundle construction lemma (Proposition 1.2.8). We only need to find the corresponding candidates for the local trivialisation maps; to this end, we use the $(\tau_{i,j})_{i,j\in I}$ to identify the future fibres with $\mathbb{R}^{\dim_{\mathbb{R}} V}$.

Let $k := \dim_{\mathbb{R}} V$. Without loss of generality we may assume that $V = \mathbb{R}^k$ (check!). For each $x \in M$, we set $E_x := \mathbb{R}^k$. We then consider $E := \bigsqcup_{x\in M} E_x$ and let $\pi\colon E \longrightarrow M$ be the index-projection map. Moreover, for each $x \in M$, we choose an $i_x \in I$ with $x \in U_{i_x}$.

For $i \in I$, we then define

$$\begin{aligned} \varphi_i\colon \pi^{-1}(U_i) &\longrightarrow U_i \times \mathbb{R}^k \\ E_x \ni v &\longmapsto \bigl(x, \tau_{i,i_x}(x)(v)\bigr). \end{aligned}$$

By construction, φ_i is bijective (check!) and φ_i is fibre-preserving and a fibre-wise $\mathbb{R}$-linear isomorphism (check!).

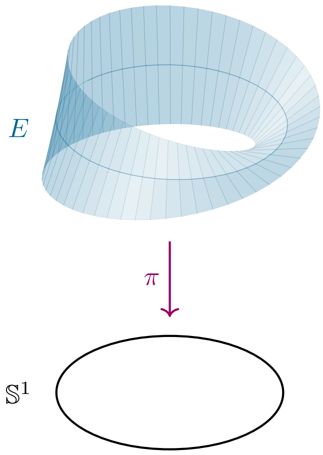


Figure 1.8.: The Möbius strip as a line bundle over $\mathbb{S}^1$

It remains to check the transition map condition: For all $i, j \in I$, all $x \in U_i \cap U_j$, and all $v \in E_x$, we have

$$\begin{aligned}
\varphi_i \circ \varphi_j^{-1}(x, v) &= \bigl(x, (\tau_{i,i_x}(x) \circ (\tau_{j,i_x}(x))^{-1})(v)\bigr) && \text{(by construction)} \\
&= \bigl(x, (\tau_{i,i_x}(x) \circ \tau_{i_x,j}(x))(v)\bigr) && \text{(Remark 1.2.10)} \\
&= \bigl(x, \tau_{i,j}(x)(v)\bigr). && \text{(Remark 1.2.10)}
\end{aligned}$$

Applying the vector bundle construction lemma proves existence.

The uniqueness claim can be derived from the uniqueness part of Proposition 1.2.8 (check!). ☐

Example 1.2.11 (Möbius strip)**.** We can view the open Möbius strip as a line bundle over the circle $\mathbb{S}^1$ (Figure 1.8). A simple way of constructing this line bundle rigorously is to use Corollary 1.2.9 (Exercise; two patches suffice).

Example 1.2.12 (non-triviality of the Möbius strip)**.** If $\pi \colon E \longrightarrow M$ is a trivial vector bundle of non-zero dimension, then there exists a *nowhere vanishing section*, i.e., a section $s \in \Gamma(\pi)$ with

$$\forall_{x \in M} \quad s(x) \neq 0$$

(check!).

One can check that the Möbius bundle over S^1 does *not* admit such a nowhere vanishing section (check!). Hence, the Möbius bundle is *not* trivial.

1.2.3 The tangent bundle

We now assemble the tangent spaces of a smooth manifold into the tangent bundle.

- To this end, we use smooth charts and their differentials as input for the glueing construction (Proposition 1.2.8).
- This construction is functorial: Smooth maps induce bundle maps between the tangent bundles.

Proposition and Definition 1.2.13 (tangent bundle)**.** *Let M be a smooth manifold of dimension n with smooth structure A.*

- *For a smooth chart $(\varphi\colon U \to U') \in A$, we set*
$$\begin{aligned}\widetilde{\varphi}\colon \bigsqcup_{x\in U} \mathrm{T}_x M &\longrightarrow U \times \mathbb{R}^n\\ \mathrm{T}_x M \ni v &\longmapsto \bigl(x, \mathrm{d}_x \varphi(v)\bigr).\end{aligned}$$
- *For smooth charts $(\varphi\colon U \to U'), (\psi\colon V \to V') \in A$, we set*
$$\begin{aligned}\tau_{\varphi,\psi}\colon U \cap V &\longrightarrow \mathrm{GL}(\mathbb{R}^n)\\ x &\longmapsto \mathrm{d}_{\psi(x)}(\varphi \circ \psi^{-1}).\end{aligned}$$

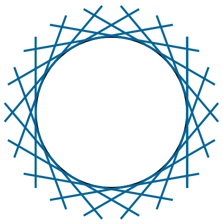
Then the families $(\widetilde{\varphi})_{\varphi\in A}$ and $(\tau_{\varphi,\psi})_{\varphi,\psi\in A}$ satisfy the conditions of Proposition 1.2.8. The corresponding smooth vector bundle with the fibres $(\mathrm{T}_x M)_{x\in M}$ is the tangent bundle $\mathrm{T}\, M \longrightarrow M$ *of M.*

Proof. This is a straightforward computation:

Fibrewise bijectivity and linearity of the $\widetilde{\varphi}$. If $\varphi\colon U \longrightarrow U'$ is a smooth chart of M, then for every $x \in U$, the $\mathbb{R}$-linear map $\mathrm{d}_x \varphi\colon T_x M \longrightarrow \mathbb{R}^n$ is an isomorphism (Why?). This shows that $\widetilde{\varphi}$, which is clearly fibre-preserving, is bijective and fibrewise linear.

Smoothness of the $\tau_{\varphi,\psi}$. Let $(\varphi\colon U \to U'), (\psi\colon V \to V') \in A$. Then, $\varphi \circ \psi^{-1}|_{\psi(U\cap V)}$ is smooth (in the sense of multi-variable analysis) and ψ is smooth (Example 1.1.19). Therefore, also the combination $\tau_{\varphi,\psi}$ is smooth.

Transition function condition. Moreover, the chain rule for differentials (Example 1.1.27) shows that
$$\widetilde{\varphi} \circ \widetilde{\psi}^{-1}(x, v) = \bigl(x, \mathrm{d}_{\psi(x)}(\varphi \circ \psi^{-1})(v)\bigr) = \bigl(x, \tau_{\varphi,\psi}(x)(v)\bigr)$$
holds for all $x \in U \cap V$ and all $v \in \mathrm{T}_x M$ (check!). □



Toggle tangent bundle

Figure 1.9.: The tangent bundle of $\mathbb{S}^1$, intuitively

Alternatively, we also could have used the cocycle condition directly (Corollary 1.2.9). However, strictly speaking, this would only define the tangent bundle up to isomorphism and would make it a bit more cumbersome to find concrete local descriptions.

Remark 1.2.14 (charts for the tangent bundle)**.** Looking at the construction in the proof of Proposition 1.2.8 shows: If M is a smooth manifold with tangent bundle $\pi\colon \mathrm{T}\,M \longrightarrow M$ and if $\varphi\colon U \longrightarrow U'$ is a smooth chart of M, then

$$\begin{aligned} \pi^{-1}(U) &\longrightarrow U' \times \mathbb{R}^{\dim M} \\ (x, v) &\longmapsto \bigl(\varphi(x), \mathrm{d}_x\,\varphi(v)\bigr) \end{aligned}$$

is a smooth chart of $\mathrm{T}\,M$ (check!).

Quick check 1.2.15. Let M be a smooth manifold of dimension n. Which dimension does the smooth manifold $\mathrm{T}\,M$ have?
Hint!

Example 1.2.16 (tangent bundle of the circle)**.** The tangent bundle $\mathrm{T}\,\mathbb{S}^1$ of the circle $\mathbb{S}^1$ is trivial: This fact can be visualised as in Figure 1.9; however, it should be noted that such pictures of tangent bundles can be deceiving.

In order to prove this, we show that the transition functions associated with suitable smooth charts are constant:

- For $a \in \mathbb{R}$, the inverse map φ_a of

$$\begin{aligned} (a, a + 2\pi) &\longrightarrow \mathbb{S}^1 \setminus \{(\cos a, \sin a)\} \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

  is a smooth chart for $\mathbb{S}^1$ (check!). Moreover, the domains of all these charts cover $\mathbb{S}^1$.

- For all $x \in \mathbb{S}^1$ and all $a, b \in \mathbb{R}$, we have

$$\mathrm{d}_{\varphi_b(x)}(\varphi_a \circ \varphi_b^{-1}) = \boxed{?}$$

Therefore, the local trivialisations glue to a trivialisation on all of $\mathbb{S}^1$ and we can deduce that the tangent bundle of $\mathbb{S}^1$ is trivial (check!).

In contrast, one can show that the tangent bundle of $\mathbb{S}^2$ is *not* trivial and that the (non-)triviality of tangent bundles of spheres is related to real division algebras [24].

Proposition and Definition 1.2.17 (the differential). *Let $f\colon M \longrightarrow N$ be a smooth map between smooth manifolds. Then*

$$\begin{aligned} \mathrm{d}\, f\colon \mathrm{T}\, M &\longrightarrow \mathrm{T}\, N \\ \mathrm{T}_x\, M \ni v &\longmapsto \mathrm{d}_x\, f(v) \end{aligned}$$

is a bundle map (over f). We call $\mathrm{d}\, f$ the differential *of f.*

Proof. By construction, $\mathrm{d}\, f$ is fibre-preserving and fibrewise $\mathbb{R}$-linear.

Thus, it remains to check that $\mathrm{d}\, f$ is smooth. With respect to smooth charts on $\mathrm{T}\, M$ and $\mathrm{T}\, N$ induced by smooth charts $\varphi\colon U \longrightarrow U'$ and $\psi\colon V \longrightarrow V'$ of M and N (Remark 1.2.14), the map $\mathrm{d}\, f$ locally is of the form

$$\begin{aligned} U' \times \mathbb{R}^{\dim M} &\longrightarrow V' \times \mathbb{R}^{\dim N} \\ (x, v) &\longmapsto \boxed{?} \end{aligned}$$

which is smooth (check!). Therefore, $\mathrm{d}\, f\colon \mathrm{T}\, M \longrightarrow \mathrm{T}\, N$ is smooth. ☐

Remark 1.2.18 (the definitive version of the chain rule). We can now reinterpret the well-known chain rule for differentials in a conceptual way: The tangent bundle defines a functor F from the category Mfd of smooth manifolds and smooth maps to the category VectB of smooth vector bundles:

- On objects: If M is a smooth manifold, then we set $F(M) := \mathrm{T}\, M$.
- On morphisms: If M and N are smooth manifolds and $f \in C^\infty(M, N)$, then we set $F(f) := \mathrm{d}\, f$.

That F is a functor means in explicit terms that (Appendix A.1):

- For all smooth manifolds M, we have $F(\mathrm{id}_M) = \mathrm{id}_M$.
- For all smooth maps $f\colon M \longrightarrow N$ and $g\colon N \longrightarrow P$ between smooth manifolds, we have the *chain rule*

$$F(g \circ f) = F(g) \circ F(f).$$

The fact that F indeed does have these properties is a consequence of the corresponding fibrewise properties (Example 1.1.27).

1.2.4 Tensor bundles

Differential geometry works with vector fields (i.e., sections of the tangent bundle) as well as with various types of tensor fields to model multi-variable constructs such as Riemannian metrics or curvature tensors. Tensor fields arise as follows:

- Tensor fields will be sections of tensor bundles.
- These tensor bundles are constructed by applying functors from linear algebra to the fibres of the tangent bundle.
- Looking at the (cocycle) construction of vector bundles shows that we should impose a smoothness condition on the functor to obtain *smooth* cocycles out of smooth cocycles.

In order to arrive quickly at local coordinates for the tensor bundles, we will use the more explicit construction from Proposition 1.2.8 instead of the cocycle-only description.

Definition 1.2.19 (smooth functor)**.** Let $\mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}$ denote the category of all finite-dimensional $\mathbb{R}$-vector spaces. A functor $F\colon \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \longrightarrow \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}$ is *smooth* if for all finite-dimensional vector spaces V and W, the map

$$\begin{aligned} \mathrm{Hom}_{\mathbb{R}}(V, W) &\longrightarrow \mathrm{Hom}_{\mathbb{R}}\bigl(F(V), F(W)\bigr) \\ f &\longmapsto F(f) \end{aligned}$$

is smooth (where the finite-dimensional Hom-spaces carry the standard smooth structure; Example 1.1.8). Analogously, we introduce the notion of contravariant smooth functors.

Remark 1.2.20 (smooth functors and automorphisms)**.** Let $F\colon \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \longrightarrow \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}$ be a smooth functor. If V is a finite-dimensional $\mathbb{R}$-vector space, then F induces a well-defined map

$$\begin{aligned} \mathrm{GL}(V) &\longrightarrow \mathrm{GL}\bigl(F(V)\bigr) \\ f &\longmapsto F(f) \end{aligned}$$

(check!) that is smooth (check!).

Example 1.2.21 (smooth functors)**.**

- Dualising vector spaces yields a contravariant functor

$$\cdot^* := \mathrm{Hom}_{\mathbb{R}}(\,\cdot\,, \mathbb{R})\colon \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \longrightarrow \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}\,.$$

  This functor is smooth: If V and W are finite-dimensional vector spaces, then the induced map

$$\begin{aligned}\mathrm{Hom}_{\mathbb{R}}(V,W) &\longrightarrow \mathrm{Hom}_{\mathbb{R}}(W^*,V^*)\\ f &\longrightarrow f^* := (g \mapsto g \circ f)\end{aligned}$$

is $\mathbb{R}$-linear. As the domain and target spaces are finite-dimensional $\mathbb{R}$-vector spaces, this map is smooth (Example 1.1.27).

- The tensor square functor $\mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \longrightarrow \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}$ is defined
  - on objects by $V \longmapsto V \otimes_{\mathbb{R}} V$, and
  - on morphisms by

$$\begin{aligned}\mathrm{Hom}_{\mathbb{R}}(V,W) &\longmapsto \mathrm{Hom}_{\mathbb{R}}(V \otimes_{\mathbb{R}} V, W \otimes_{\mathbb{R}} W)\\ f &\longmapsto f \otimes_{\mathbb{R}} f := (v \otimes w \mapsto f(v) \otimes f(w))\end{aligned}$$

  for all finite-dimensional $\mathbb{R}$-vector spaces V and W. Rewriting this map in terms of bases on V and W shows that the corresponding coefficient functions are polynomials in multiple variables (check!). Therefore, this is a smooth map.

  Hence, the tensor square functor is smooth.

Proposition 1.2.22 (applying smooth functors to smooth vector bundles)**.** *Let $F\colon \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \longrightarrow \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}$ be a covariant smooth functor. Then F induces a functor* $\mathsf{VectB} \longrightarrow \mathsf{VectB}$ *as follows:*

1. *Let M be a smooth manifold, let $\pi\colon E \longrightarrow M$ be a smooth vector bundle over M of rank $k \in \mathbb{N}$, and let A be the set of all local trivialisations of π.*
   - *For $(\varphi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k) \in A$, we set*

$$\begin{aligned}\widetilde{\varphi}\colon \bigsqcup_{x\in U} F(E_x) &\longrightarrow U \times F(\mathbb{R}^k)\\ F(E_x) \ni v &\longmapsto \bigl(x, F(\varphi_x)(v)\bigr),\end{aligned}$$

     where $\varphi_x\colon E_x \longrightarrow \mathbb{R}^k$ is given by $w \longmapsto \mathrm{pr}_{\mathbb{R}^k} \circ \varphi(w)$.
   - *For $(\varphi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k), (\psi\colon \pi^{-1}(V) \to V \times \mathbb{R}^k) \in A$, we set*

$$\begin{aligned}\widetilde{\tau}_{\varphi,\psi}\colon U \cap V &\longrightarrow \mathrm{GL}\bigl(F(\mathbb{R}^k)\bigr)\\ x &\longmapsto F\bigl(\tau_{\varphi,\psi}(x)\bigr),\end{aligned}$$

     where $\tau_{\varphi,\psi}$ is the transition function between φ and ψ for π.

   Then the families $(\widetilde{\varphi})_{\varphi\in A}$ and $(\widetilde{\tau}_{\varphi,\psi})_{\varphi,\psi\in A}$ satisfy the conditions of Proposition 1.2.8. We denote the resulting smooth vector bundle with fibres $(F(E_x))_{x\in M}$ by $F(\pi)$.

2. *Moreover, F can be applied fibrewise to bundle maps between smooth vector bundles, which results in bundle maps.*

Proof. Ad 1. Strictly speaking, we aim at applying Proposition 1.2.8 to trivialisations to the $\mathbb{R}$-vector space $F(\mathbb{R}^k)$ instead of to the fibre $\mathbb{R}^m$ with $m := \dim_{\mathbb{R}} F(\mathbb{R}^k)$. Choosing an $\mathbb{R}$-isomorphism $F(\mathbb{R}^k) \cong_{\mathbb{R}} \mathbb{R}^m$ once and for all, this is independent of the base points; therefore, this modification can be easily made.

We then only need to check the conditions, which is a straightforward computation:

- Because F is a functor, the family $(\widetilde{\varphi})_{\varphi \in A}$ satisfies (fibrewise) bijectivity and fibrewise linearity (check!).
- As the functor F is smooth, the maps $(\widetilde{\tau}_{\varphi,\psi})_{\varphi,\psi \in A}$ are smooth (Remark 1.2.20).
- Moreover, by construction and functoriality of F, the transition function condition is satisfied (check!).

Ad 2. This can be checked locally; again, smoothness of F ensures that the resulting map between the total spaces is smooth (check!). □

In order to deal with mixed tensor fields, we need two extensions of this principle:

Remark 1.2.23 (applying smooth functors to smooth vector bundles, contravariant case)**.** Similarly to Proposition 1.2.22, we can also transform bundles by applying contravariant smooth functors. There is only one small technical issue with the transition functions. Which one?

Thus, in the contravariant case, we modify the construction of the transition functions to

$$\begin{aligned} \widetilde{\tau}_{\varphi,\psi} \colon U \cap V &\longrightarrow \mathrm{GL}\bigl(F(\mathbb{R}^k)\bigr) \\ x &\longmapsto \Bigl(F\bigl(\tau_{\varphi,\psi}(x)\bigr)\Bigr)^{-1}. \end{aligned}$$

For example, if π is a smooth vector bundle over a smooth manifold M, then we can construct the dual vector bundle π^* over M; this construction is functorial (contravariant).

Remark 1.2.24 (applying smooth functors in two variables to smooth vector bundles)**.** Functors in two vector space variables can be modelled as functors $\mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \times \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \longrightarrow \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}$. For functors $\mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \times \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \longrightarrow \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}$, we can introduce a notion of smoothness, similar to the one in Definition 1.2.19 (check!).

In analogy with Proposition 1.2.22, we can then construct new vector bundles by applying such a smooth functor to two smooth vector bundles on the same manifold.

For example, the tensor product functor $\cdot \otimes_{\mathbb{R}} \cdot : \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \times \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \longrightarrow \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}$ is smooth (check!) and so we can construct tensor products $\pi \otimes_{\mathbb{R}} \pi'$ of smooth vector bundles π and π' over the same base manifold; this construction is functorial (in both variables).

Finally, we specialise these general constructions to the case of the tangent bundle:

Definition 1.2.25 (tensor bundles)**.** Let M be a smooth manifold and let $\pi\colon \mathrm{T}\, M \longrightarrow M$ be its tangent bundle. Moreover, let $k, \ell \in \mathbb{N}$. We then write $\mathrm{T}^*\, M := \pi^*$ for the *cotangent bundle* and

$$\mathbf{T}^{k,\ell}\, M := \mathbf{T}^{k,l}(\pi) := (\underbrace{\pi \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \pi}_{k \text{ times}} \otimes_{\mathbb{R}} \underbrace{\pi^* \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \pi^*}_{\ell \text{ times}})^*.$$

The space of sections of this vector bundle is denoted by $\Gamma(\mathbf{T}^{k,l}\, M)$; these sections are called *k-covariant and ℓ-contravariant tensor fields on M* or *(k,ℓ)-tensor fields on M*.

Caveat 1.2.26 (mixed variance)**.** The mixed tensor bundles over a smooth manifold are constructed from components of different variance. Therefore, in general, we cannot expect these constructions to be functorial on the category of all smooth manifolds.

Moreover, the terminology of covariance and contravariance in the setting of tensor bundles might look counterintuitive from the functorial point of view. However, this language is adapted to the change of coordinates after base change.

Remark 1.2.27 (some important identifications)**.** Let V be a finite-dimensional $\mathbb{R}$-vector space. For $k, \ell \in \mathbb{N}$, we define

$$\mathbf{T}^{k,\ell}\, V := (\underbrace{V \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V}_{k \text{ times}} \otimes_{\mathbb{R}} \underbrace{V^* \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V^*}_{\ell \text{ times}})^*.$$

We then have the following canonical (i.e., independent of the choice of bases) isomorphisms of $\mathbb{R}$-vector spaces:

- $\mathbf{T}^{k,0}\, V \cong_{\mathbb{R}} (V^{\otimes k})^* \cong_{\mathbb{R}} (V^*)^{\otimes k}$
- $\mathbf{T}^{0,\ell}\, V \cong_{\mathbb{R}} (V^{*\otimes \ell})^* \cong (V^{\otimes \ell})^{**} \cong_{\mathbb{R}} V^{\otimes \ell}$
- $\mathbf{T}^{1,0}\, V \cong_{\mathbb{R}} V^*$
- $\mathbf{T}^{0,1}\, V \cong_{\mathbb{R}} V^{**} \cong_{\mathbb{R}} V$
- $\mathbf{T}^{0,0}\, V \cong_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R}) \cong_{\mathbb{R}} \mathbb{R}$
- $\mathbf{T}^{1,1}\, V \cong_{\mathbb{R}} (V \otimes_{\mathbb{R}} V^*)^* \cong_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}}(V, V)$
- $\mathbf{T}^{k,\ell}\, V \cong_{\mathbb{R}} (V^{\otimes k} \otimes_{\mathbb{R}} V^{*\otimes \ell})^* \cong_{\mathbb{R}} (V^*)^{\otimes k} \otimes_{\mathbb{R}} V^{**\otimes \ell} \cong_{\mathbb{R}} (V^*)^{\otimes k} \otimes_{\mathbb{R}} V^{\otimes \ell}$

These canonical isomorphisms are constructed from the following canonical isomorphisms (for finite-dimensional $\mathbb{R}$-vector spaces V and W):

$$\begin{aligned}
V^* \otimes_\mathbb{R} W^* &\longrightarrow (V \otimes_\mathbb{R} W)^* \\
f \otimes g &\longmapsto \text{Hint!} \\
V &\longrightarrow V^{**} \\
v &\longmapsto \text{Hint!} \\
V^{\otimes 0} &\longmapsto \mathbb{R} \\
() &\longmapsto \text{Hint!} \\
\operatorname{Hom}_\mathbb{R}(V, V) &\longrightarrow (V \otimes_\mathbb{R} V^*)^* \\
f &\longmapsto \text{Hint!}
\end{aligned}$$

However, it should be noted that there is *no* canonical isomorphism $V \cong_\mathbb{R} V^*$!

Because the above isomorphisms are canonical, they induce corresponding canonical isomorphisms for the tensor bundles over smooth manifolds (check!). In particular, $\Gamma(\mathrm{T}^{0,0} M) \cong_\mathbb{R}$? .

Remark 1.2.28 (tensor fields, explicitly)**.** How can we describe tensor fields explicitly and perform computations with them (at least locally)? Let M be a smooth manifold of dimension $n \in \mathbb{N}$ with tangent bundle $\pi\colon \mathrm{T}M \longrightarrow M$ and let $\varphi\colon U \longrightarrow U'$ be a smooth chart for M. We denote the corresponding (smooth) coordinate functions by

$$x^1, \dots, x^n \colon U \longrightarrow \mathbb{R}.$$

Then

$$(\mathrm{d}\, x^1, \dots, \mathrm{d}\, x^n)$$

(viewed as functions $x \mapsto \mathrm{d}_x\, x^j$) is a *local frame* for π^* on U, i.e., these functions are smooth and for each $x \in U$, the tuple $(\mathrm{d}_x\, x^1, \dots, \mathrm{d}_x\, x^n)$ is a basis for $(\mathrm{T}_x M)^*$.

For $x \in U$, let $(E_1(x), \dots, E_n(x))$ be the dual basis to $(\mathrm{d}_x\, x^1, \dots, \mathrm{d}_x\, x^n)$. Then, $(E_1, \dots, E_n)$ is a local frame for π^{**} on U and whence (under the canonical isomorphism $\pi^{**} \cong_{\mathsf{VectB}} \pi$) a local frame for π (Figure 1.10).

Local frames of $\mathrm{T}M$ and $\mathrm{T}^* M$ that arise in this way from charts (i.e., from local coordinates) are also called *coordinate frames*.

Now, let $k, \ell \in \mathbb{N}$. Then the fibrewise dual basis of

$$\bigl(E_{i_1} \otimes \dots \otimes E_{i_k} \otimes \mathrm{d}\, x^{j_1} \otimes \dots \otimes \mathrm{d}\, x^{j_\ell}\bigr)_{i_1,\dots,i_k,j_1,\dots,j_\ell \in \{1,\dots,n\}}$$

is a local frame for $\mathbf{T}^{k,\ell} M$ on U. In view of the canonical isomorphisms from Remark 1.2.27 and the duality relation between the two local frames on $\mathrm{T}M$ and $\mathrm{T}^* M$, we obtain that

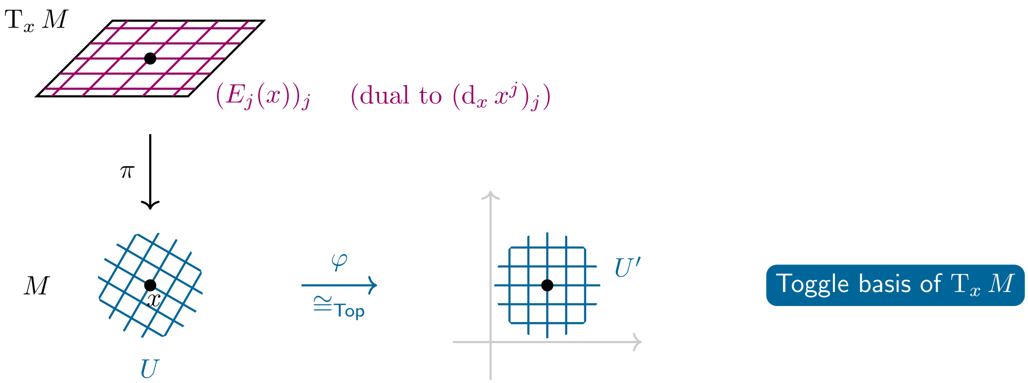


Figure 1.10.: Coordinate frames, schematically

$$\big(\mathrm{d}\,x^{i_1} \otimes \cdots \otimes \mathrm{d}\,x^{i_k} \otimes E_{j_1} \otimes \cdots \otimes E_{j_\ell}\big)_{i_1,\dots,i_k,j_1,\dots,j_\ell \in \{1,\dots,n\}}$$

can be viewed as a local frame for $\mathbf{T}^{k,\ell}\, M$ on U.

Therefore, sections on U of $\mathbf{T}^{k,\ell}\, M$ correspond bijectively to families

$$(f^{i_1,\dots,i_k}_{j_1,\dots,j_\ell})_{i_1,\dots,i_k,j_1,\dots,j_\ell \in \{1,\dots,n\}}$$

of functions in $C^\infty(U)$. More explicitly, if $f \in \Gamma(\mathrm{T}^{k,\ell}\, M)$, then there exists such a family that allow to express f in local coordinates via

$$\forall_{x \in U} \quad f(x) = \underbrace{f^{i_1,\dots,i_k}_{j_1,\dots,j_\ell}(x) \cdot \mathrm{d}\,x^{i_1} \otimes \cdots \otimes \mathrm{d}\,x^{i_k} \otimes E_{j_1} \otimes \cdots \otimes E_{j_\ell}} \qquad \text{(multi-Einstein summation!)}$$

$$= \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_\ell=1}^{n} f^{i_1,\dots,i_k}_{j_1,\dots,j_\ell}(x) \cdot \mathrm{d}\,x^{i_1} \otimes \cdots \otimes \mathrm{d}\,x^{i_k} \otimes E_{j_1} \otimes \cdots \otimes E_{j_\ell}.$$

Concrete examples of this principle will be discussed in Example 1.3.9.

For the sake of completeness, we briefly mention how differential forms fit into this setup:

Remark 1.2.29 (differential forms)**.** Let M be a smooth manifold and let $k \in \mathbb{N}$. Then the k-th exterior product $\bigwedge^k \colon \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}} \longrightarrow \mathsf{Vect}^{\mathsf{fin}}_{\mathbb{R}}$ is a smooth functor (check!). We write $\bigwedge^k(\mathrm{T}\,M)$ for the k-th exterior product of the tangent bundle of M. The sections of the dual bundle $(\bigwedge^k(\mathrm{T}\,M))^*$ are called *smooth differential k-forms on M* and we write

$$\Omega^k M := \Gamma\big(\big(\textstyle\bigwedge^k(\mathrm{T}\,M)\big)^*\big).$$

By the universal property of exterior products, differential k-forms on M consist of alternating k-forms on $\mathrm{T}_x M$ for each $x \in M$.

Our main interest in differential forms will be in connection with orientations and the volume form (Chapter 1.5.3).

Outlook 1.2.30 (K-theory)**.** Let M be a smooth manifold. Using smooth functors in two variables (Remark 1.2.24), we can introduce the direct sum $\pi \oplus \pi'$ of smooth vector bundles π, π' over M. The set of isomorphism classes of smooth vector bundles over M forms a commutative monoid with respect to this direct sum. Taking the Grothendieck construction of this monoid (i.e., adjoining formal inverses) leads to an Abelian group, the *(smooth) topological K-theory group* $K(M)$ of M [2]. This K-theory group $K(M)$ is related to the algebraic K-theory of the ring $C^\infty(M)$ (in degree 0) [31].

For example, in $K(S^1)$ the sum of (class of) the Möbius strip bundle with itself results in the class of the trivial bundle of rank 2 over M (check!).

1.3 Riemannian manifolds

In the linear algebraic approach to traditional Euclidean geometry, geometric concepts, such as lengths and angles, are described in terms of inner products.

We now add the Riemannian structure to smooth manifolds. A Riemannian manifold is a smooth manifold that is locally an inner product space. We will formalise this idea using Riemannian metrics, which are "smoothly parametrised" inner products on the tangent spaces.

This structure will then allow us to generalise the length of curves, distances, angles, volumes, and other geometric notions from Euclidean spaces to Riemannian manifolds (Chapter 1.5). In the present section, we will first establish foundations of Riemannian metrics.

1.3.1 Riemannian metrics

Inner products are covariant 2-tensors that satisfy symmetry and positive definiteness. Hence, a Riemannian metric on a smooth manifold will be defined as a section of the bundle of covariant 2-tensors that satisfies symmetry and positive definiteness at each point:

Definition 1.3.1 (Riemannian metric)**.** Let M be a smooth manifold. A *Riemannian metric* is a section $g \in \Gamma(\mathbf{T}^{2,0} M)$ with the following properties:

- *Symmetry.* For every $x \in M$, we have

$$\forall_{v,w \in \mathrm{T}_x M} \quad g_x(v \otimes w) = g_x(w \otimes v).$$

- *Positive definiteness.* For every $x \in M$, we have

$$\forall_{v \in \mathrm{T}_x M \setminus \{0\}} \quad g_x(v \otimes v) > 0.$$

Here, we denoted the value of g at x by g_x. Alternatively, it is also common to write $\langle\,\cdot\,,\,\cdot\,\rangle_x^g$ (Quick check 1.3.2).

We write $\operatorname{Riem}(M)$ for the set of all Riemannian metrics on M.

Quick check 1.3.2. Inner products are also required to be bilinear. Where is this encoded in the definition of Riemannian metrics?
Hint

Thus, if g is a Riemannian metric on M, then for each $x \in M$, the element $g_x \in (\mathrm{T}_x M \otimes_{\mathbb{R}} \mathrm{T}_x M)^*$ indeed can be viewed as an inner product on $\mathrm{T}_x M$:

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle_x^g \colon \mathrm{T}_x M \times \mathrm{T}_x M &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto \langle v, w\rangle_x^g := g_x(v \otimes w). \end{aligned}$$

The prototypical example of a Riemannian manifold is Euclidean space with the Euclidean inner product. Using charts, we can pull back this model case to more general smooth manifolds. In combination with partitions of unity, we will thus obtain that every smooth manifold admits a Riemannian metric (Theorem 1.3.6).

Example 1.3.3 (Euclidean space: Riemannian metric)**.** Let $n \in \mathbb{N}$. Then the map

$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathrm{T}^{2,0}\,\mathbb{R}^n \\ x &\longmapsto \text{standard Euclidean inner product } \langle\,\cdot\,,\,\cdot\,\rangle \text{ on } \mathbb{R}^n \end{aligned}$$

is a Riemannian metric on $\mathbb{R}^n$, the *Euclidean Riemannian metric on* $\mathbb{R}^n$.

More generally: If $U \subset \mathbb{R}^n$ is an open subset, carrying the induced smooth structure, then

$$\begin{aligned} U &\longrightarrow \mathrm{T}^{2,0}\,U \cong_{\mathbb{R}} \mathrm{T}^{2,0}\,\mathbb{R}^n \text{ (canonical isomorphism!)} \\ x &\longmapsto \text{standard Euclidean inner product } \langle\,\cdot\,,\,\cdot\,\rangle \text{ on } \mathbb{R}^n \end{aligned}$$

is a Riemannian metric on U, the *Euclidean Riemannian metric on* U.

Proposition 1.3.4 (pulling back Riemannian metrics)**.** *Let $f \colon M \longrightarrow N$ be a local diffeomorphism between smooth manifolds and let g be a Riemannian metric on N. Then*

$$\begin{aligned} f^*g \colon M &\longrightarrow \mathbf{T}^{2,0}\,M \\ x &\longmapsto \bigl(v \otimes w \mapsto g_{f(x)}(\mathrm{d}_x f(v) \otimes \mathrm{d}_x f(w))\bigr) \end{aligned}$$

is a Riemannian metric on M, the pullback of g via f.

Recall that a smooth map $f\colon M \longrightarrow N$ between smooth manifolds is a *local diffeomorphism* if for each $x \in M$, there exists an open neighbourhood $U \subset M$ of x in M such that $f(U) \subset N$ is open and $f|_U\colon U \longrightarrow f(U)$ is a diffeomorphism.

Proof. Because f is smooth and g is a smooth section of $\mathbf{T}^{2,0} N$, the composition f^*g is a smooth section of $\mathbf{T}^{2,0} M$; smoothness can, for instance, be proved rigorously via local coordinates of the tensor bundles (Remark 1.2.28). Hence, $f^*g \in \Gamma(\mathbf{T}^{2,0} M)$.

It remains to check symmetry and positive definiteness; we can do this pointwise: Let $x \in M$ and $v, w \in \mathrm{T}_x M$.

Symmetry. We have

$$\begin{aligned} f^*g_x(v \otimes w) &= g_{f(x)}\big(\mathrm{d}_x f(v) \otimes \mathrm{d}_x f(w)\big) \\ &= g_{f(x)}\big(\mathrm{d}_x f(w), \mathrm{d}_x f(v)\big) && \text{(symmetry of } g_{f(x)}\text{)} \\ &= f^*g_x(w \otimes v). \end{aligned}$$

Positive definiteness. If $v \neq 0$, then

$$f^*g_x(v \otimes v) = g_{f(x)}\big(\mathrm{d}_x f(v), \mathrm{d}_x f(v)\big).$$

Because f is a local diffeomorphism, the differential $\mathrm{d}_x f\colon \mathrm{T}_x M \longrightarrow \mathrm{T}_{f(x)} N$ is an $\mathbb{R}$-linear isomorphism (Why?). Therefore, $\mathrm{d}_x f(v) \neq 0$ and so positive definiteness of $g_{f(x)}$ implies that

$$f^*g_x(v \otimes v) = g_{f(x)}\big(\mathrm{d}_x f(v), \mathrm{d}_x f(v)\big) > 0.$$ □

Proposition 1.3.5 (scaling Riemannian metrics)**.** *Let M be a smooth manifold, let $g_1, g_2 \in \mathrm{Riem}(M)$, and let $f_1, f_2 \in C^\infty(M, \mathbb{R}_{\geq 0})$ with the property that $f_1 + f_1 > 0$ (pointwise). Then*

$$\begin{aligned} f_1 \cdot g_1 + f_2 \cdot g_2 \colon M &\longrightarrow \mathbf{T}^{2,0} M \\ x &\longmapsto \big(v \otimes w \mapsto f_1(x) \cdot (g_1)_x(v \otimes w) + f_2(x) \cdot (g_2)_x(v \otimes w)\big) \end{aligned}$$

is a Riemannian metric on M.

Proof. This can be shown similarly to Proposition 1.3.4 (Exercise). □

Theorem 1.3.6 (existence of Riemannian metrics)**.** *Let M be a smooth manifold. Then there exists a Riemannian metric on M.*

Proof. Locally we use the Euclidean Riemannian metric; we then glue these inner products via a partition of unity (Appendix A.2):

Let $(\varphi_i\colon U_i \to U_i')_{i \in I}$ be a family of charts that covers M.

- *Local situation.* For each $i \in I$, the chart $\varphi_i \colon U_i \longrightarrow U_i'$ is a diffeomorphism (Example 1.1.19). If g_i denotes the Euclidean Riemannian metric on U_i' (Example 1.3.3), then the pullback $\varphi_i^* g_i$ is a Riemannian metric on U_i (Proposition 1.3.4).

- *Global glueing.* Let $(\psi_i)_{i \in I}$ be a partition of unity of M subordinate to $(U_i)_{i \in I}$ (Theorem A.2.1). Then

$$\sum_{i \in I} \psi_i \cdot \varphi_i^* \cdot g_i$$

  is a Riemannian metric on M (Proposition 1.3.5); strictly speaking, Proposition 1.3.5 is not directly applicable to this situation, but the same arguments work (check!).

In particular, M admits a Riemannian metric. □

Corollary 1.3.7 (non-uniqueness of Riemannian metrics)**.** *Let M be a non-empty smooth manifold of non-zero dimension. Then* $\operatorname{Riem}(M)$ *is uncountable.*

Proof. We use a scaling argument: By Theorem 1.3.6, the set $\operatorname{Riem}(M)$ is non-empty and moreover $C^\infty(M, \mathbb{R}_{>0})$ acts non-trivially on $\operatorname{Riem}(M)$ (Proposition 1.3.5 and $\dim M \neq 0$). Because M is non-empty, the $\mathbb{R}$-vector space $C^\infty(M)$ is non-trivial (constant functions!), whence uncountable. Therefore, $\operatorname{Riem}(M)$ is uncountable as well. □

Remark 1.3.8 (Riemannian metrics, explicitly)**.** Let M be a smooth manifold of dimension $n \in \mathbb{N}$, let $\varphi \colon U \longrightarrow U'$ be a smooth chart of M, and let $(x^1 \colon U \to \mathbb{R}, \dots, x^n \colon U \to \mathbb{R})$ be the corresponding coordinate functions.

If g is a Riemannian metric on M, then using the associated local frame of $\mathbf{T}^{2,0} M$ on U (Remark 1.2.28), we can write

$$\begin{aligned} g|_U &= g_{i,j} \cdot \mathrm{d}\, x^i \otimes \mathrm{d}\, x^j && \text{(Einstein summation over } i \text{ and } j\text{)} \\ &= \sum_{i=1}^n \sum_{j=1}^n g_{i,j} \cdot \mathrm{d}\, x^i \otimes \mathrm{d}\, x^j \end{aligned}$$

with $g_{i,j} \in C^\infty(U)$ for all $i, j \in \{1, \dots, n\}$; moreover, for each $x \in U$, the matrix $(g_{i,j}(x))_{i,j} \in M_{n \times n}(\mathbb{R})$ is symmetric and positive definite (check!). Conversely, such local data determines a Riemannian metric on U.

Example 1.3.9 (Riemannian metrics)**.**

- For $n \in \mathbb{N}$, the Euclidean Riemannian metric (Example 1.3.3) is described in local coordinates (Remark 1.3.8) by

$$
\begin{aligned}
\underbrace{\delta_{ij} \cdot \mathrm{d}\, x^i \otimes \mathrm{d}\, x^j}_{} &= \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \cdot \mathrm{d}\, x^i \otimes \mathrm{d}\, x^j \\
&= \sum_{i=1}^{n} \mathrm{d}\, x^i \otimes \mathrm{d}\, x^i;
\end{aligned}
$$

this can be seen via evaluation on the standard basis and the identifications of Remark 1.2.27 (check!).

- The smooth section

$$\frac{1}{(x^2)^2 + 1} \cdot (\mathrm{d}\, x^1 \otimes \mathrm{d}\, x^1 + \mathrm{d}\, x^2 \otimes \mathrm{d}\, x^2)$$

of $\mathbf{T}^{2,0}\,\mathbb{R}^2$ is a Riemannian metric on $\mathbb{R}^2$ (Proposition 1.3.5). In this Riemannian metric, at points farther away from the first coordinate axis (i.e., at points with large x^2-coordinate), tangent vectors seem to be shorter than at points closer to the first coordinate axis.

- The smooth sections

$$
\begin{aligned}
\mathrm{d}\, x^1 \otimes \mathrm{d}\, x^1 - \mathrm{d}\, x^2 \otimes \mathrm{d}\, x^2 \\
\mathrm{d}\, x^1 \otimes \mathrm{d}\, x^2 + e^{x^1} \cdot \mathrm{d}\, x^2 \otimes \mathrm{d}\, x^1
\end{aligned}
$$

of $\mathbf{T}^{2,0}\,\mathbb{R}^2$ are *no* Riemannian metrics on $\mathbb{R}^2$ (Why?
).

In the literature, one also finds the multiplicative notation "$\mathrm{d}\, x^1 \cdot \mathrm{d}\, x^2$" instead of the tensor notation "$\mathrm{d}\, x^1 \otimes \mathrm{d}\, x^2$", etc..

1.3.2 Riemannian manifolds

We can now introduce Riemannian manifolds as smooth manifolds equipped with a Riemannian metric. As in other metric situations, there are several choices of "structure-preserving" maps. Particularly important cases are local isometries and isometries. However, the connection with actual metric geometry will only be established later (Chapter 1.5.2).

Riemannian manifolds are *the* objects of study in Riemannian geometry!

The Key Definition 1.3.10 (Riemannian manifold)**.** A *Riemannian manifold* is a pair (M, g), consisting of a smooth manifold M and a Riemannian metric $g \in \Gamma(\mathbf{T}^{2,0}(M))$ on M.

Example 1.3.11 (Euclidean space)**.** Let $n \in \mathbb{N}$. Then the Euclidean Riemannian metric (Example 1.3.3) turns $\mathbb{R}^n$ into a Riemannian manifold. Unless specified otherwise, we will always equip $\mathbb{R}^n$ with this Riemannian metric.

Figure 1.11.: A round sphere and a bumpy sphere, schematically

Example 1.3.12 (first fundamental form)**.** Let $N \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a smooth submanifold. Then the standard Euclidean inner product on $\mathbb{R}^N$ restricts to an inner product of the geometric tangent spaces of M (check!). Using the identification of Remark 1.1.31, we then obtain a Riemannian metric on M (smoothness can, for instance, be established in local coordinates; check!). This Riemannian metric is also referred to as the *first fundamental form* of the submanifold $M \subset \mathbb{R}^N$.

Example 1.3.13 (round spheres)**.** Let $n \in \mathbb{N}$. Then $\mathbb{S}^n$ is a smooth submanifold of $\mathbb{R}^{n+1}$ (Example 1.1.33). Thus, the first fundamental form yields a Riemannian metric g on $\mathbb{S}^n$ (Example 1.3.12), the *round metric on* $\mathbb{S}^n$. We call $(\mathbb{S}^n, g)$ the *round sphere of dimension* n.

Example 1.3.14 (bumpy spheres)**.** Let $n \in \mathbb{N}$, let g be the round metric on $\mathbb{S}^n$ (Example 1.3.13), and let $f \in C^\infty(\mathbb{S}^n, \mathbb{R}_{>0})$. Then $f \cdot g$ (in the sense of Proposition 1.3.5) is a Riemannian metric on $\mathbb{S}^n$.

If f is a non-constant function, we can imagine $(\mathbb{S}^n, f \cdot g)$ as a "bumpy sphere" (Figure 1.11). However, it is advisable to take such pictures not too literally (because we did not specify a metric embedding into $\mathbb{R}^{n+1}$ ...).

Example 1.3.15 (product, warped product)**.** Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. We will now give examples of Riemannian metrics on the cartesian product $M := M_1 \times M_2$. For $x := (x_1, x_2) \in M_1 \times M_2$, the canonical projections $M \longrightarrow M_1$ and $M \longrightarrow M_2$ induce an isomorphism

$$\mathrm{T}_x\, M \cong_{\mathbb{R}} \mathrm{T}_{x_1}\, M_1 \oplus \mathrm{T}_{x_2}\, M_2$$

(check!). Under this isomorphism, the direct sum of the inner products $(g_1)_{x_1}$ and $(g_2)_{x_2}$ yields an inner product g_x on $\mathrm{T}_x\, M$. Moreover, $x \mapsto g_x$ is a smooth section of $\mathbf{T}^{2,0}\, M$ (this can be seen locally; check!). Thus, g is a Riemannian metric on M, the *product Riemannian metric*; this situation is also denoted as $g = g_1 \oplus g_2$.

If $f \in C^\infty(M_1, \mathbb{R}_{>0})$, we can also consider the Riemannian manifold

$$(M_1, g_1) \times_f (M_2, g_2) := \bigl(M_1 \times M_2, (x, y) \mapsto (g_1)_x \oplus f^2(x) \cdot (g_2)_y\bigr),$$

the *warped product of* (M_1, g_1) *and* (M_2, g_2) *with respect to* f.

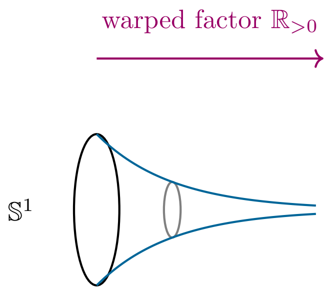


Figure 1.12.: A warped product, schematically

For example, the warped product $\mathbb{S}^1 \times_f \mathbb{R}_{>0}$ (with the round metric on $\mathbb{S}^1$ and the Euclidean Riemannian metric on $\mathbb{R}_{>0}$) with respect to the function $f \colon \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$, $t \longmapsto e^{-t}$ can be visualised roughly as cusp-like surface of revolution (Figure 1.12; the scaling of the factor corresponds to the scaling by the warping function).

Quick check 1.3.16 (products of Euclidean Riemannian metrics)**.** Let $n_1, n_2 \in \mathbb{N}$ and let g_1 and g_2 be the Euclidean Riemannian metrics on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. What is $g_1 \oplus g_2$ on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$?
Hint

Example 1.3.17 (tori)**.** Let $n \in \mathbb{N}$. Then the n-fold product $(\mathbb{S}^1)^n$ is called *n-torus*. If not specified otherwise, we will equip the n-torus with the product metric of the round metric on $\mathbb{S}^1$ (Example 1.3.13).

Definition 1.3.18 ((local) isometry)**.** Let (M, g) and (M', g') be Riemannian manifolds.

- A *local isometry* $(M, g) \longrightarrow (M', g')$ is a local diffeomorphism $f \colon M \longrightarrow M'$ with $g = f^* g'$.
- An *isometry* $(M, g) \longrightarrow (M', g')$ is a diffeomorphism $f \colon M \longrightarrow M'$ with $g = f^* g'$.

Definition 1.3.19 (isometry group)**.** Let (M, g) be a Riemannian manifold. The *isometry group* $\operatorname{Isom}(M, g)$ is the group of all Riemannian isometries $(M, g) \longrightarrow (M, g)$ with respect to composition of maps (i.e., the automorphism group of (M, g) in the category of Riemannian manifolds and local isometries).

Once we have constructed a classical metric out of the Riemannian metric, we will prove that this Riemannian isometry group indeed is the classical isometry group (Theorem 3.2.23).

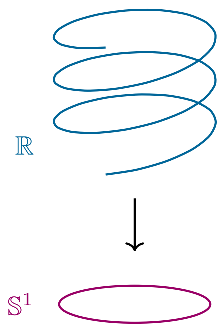


Figure 1.13.: The map $t \longmapsto (\cos t, \sin t)$

Example 1.3.20 (local isometries)**.**

- Let $n \in \mathbb{N}$ and let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an $\mathbb{R}$-linear map. Then f is an isometry of $\mathbb{R}^n$ with the Euclidean Riemannian metric if and only if f is induced (with respect to the standard basis) by an element of the orthogonal group $\mathrm{O}(n)$ (check!).

  Also, all translation maps $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ are isometries (check!).

- The map (Figure 3.1)

$$\begin{aligned} e\colon \mathbb{R} &\longrightarrow \mathbb{S}^1 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

  is a local isometry with respect to the Euclidean Riemannian metric g on $\mathbb{R}$ and the round metric g° on $\mathbb{S}^1$: For all $t \in \mathbb{R}$, we have

$$\begin{aligned} \mathrm{d}_t\, e\colon\ \mathrm{T}_t\,\mathbb{R} &\longrightarrow \mathrm{T}_{e(t)}\,\mathbb{S}^1 \\ v &\longmapsto (-v \cdot \sin t, v \cdot \cos t), \end{aligned}$$

  where we use the explicit geometric description of the tangent space of $\mathbb{S}^1$ as submanifold of $\mathbb{R}^2$ (Remark 1.1.31). Therefore, for all $v, w \in \mathrm{T}_t\,\mathbb{R}$, we obtain

$$\begin{aligned} (e^* g^\circ)_t(v \otimes w) &= g^\circ_{e(t)}\big(\mathrm{d}_t\, e(v) \otimes \mathrm{d}_t\, e(w)\big) && \text{(definition of the pullback metric)} \\ &= g^\circ_{e(t)}\big((-v \cdot \sin t, v \cdot \cos t) \otimes (-w \cdot \sin t, w \cdot \cos t)\big) && \text{(definition of } e\text{)} \\ &= v \cdot w \cdot (\sin t)^2 + v \cdot w \cdot (\cos t)^2 && \text{(first fundamental form)} \\ &= v \cdot w \cdot 1 && \text{(as } \sin^2 + \cos^2 = 1\text{)} \\ &= g_t(v \otimes w). && \text{(Euclidean Riemannian metric)} \end{aligned}$$

However, e is *not* an isometry. Why?

Outlook 1.3.21 (general relativity). General relativity is usually modelled by so-called Lorentzian manifolds; in contrast with the Riemannian case, one works with symmetric sections of $\mathbf{T}^{2,0}$ that are not positive definite, but have signature $(n-1,1)$ [3].

1.4 Model spaces

We introduce the three types of model spaces: spheres, Euclidean spaces, and hyperbolic spaces. These spaces will later serve as "models" of positive, vanishing, and negative curvature, respectively.

In addition, we will also show that these spaces carry a high degree of symmetry.

1.4.1 Homogeneous spaces

Symmetries are transformations of geometric objects that preserve the geometric structure, i.e., the elements of the isometry group. The degree of symmetry of a smooth manifold is measured in terms of transitivity properties of the action of the isometry group on the manifold and on the tangent spaces.

This will lead to homogeneous, isotropic, and symmetric manifolds (as well as the corresponding local versions).

In this section, we will only introduce the definitions; examples will be given in subsequent sections – via the model spaces.

Remark 1.4.1 (actions of the isometry group). If (M,g) is a Riemannian manifold, then the isometry group acts by function application on (M,g) (check!):

$$\begin{aligned} \operatorname{Isom}(M,g) \times M &\longrightarrow M \\ (f,x) &\longmapsto f(x) \end{aligned}$$

For $x \in M$, the stabiliser group $\operatorname{Isom}(M,g)_x := \{f \in \operatorname{Isom}(M,g) \mid f(x) = x\}$ of this action at x acts on $\mathrm{T}_x M$ via the differentials (check!), the so-called *isotropy representation of M at x*:

$$\begin{aligned} \operatorname{Isom}(M,g)_x \times \mathrm{T}_x M &\longrightarrow \mathrm{T}_x M \\ (f,v) &\longmapsto \mathrm{d}_x f(v) \end{aligned}$$

Definition 1.4.2 (homogeneous manifold)**.** A Riemannian manifold (M,g) is *homogeneous*, if the action of the isometry group $\operatorname{Isom}(M,g)$ on (M,g) is transitive, i.e., if

$$\forall_{x,y\in M} \quad \exists_{f\in \operatorname{Isom}(M,g)} \quad f(x)=y.$$

Definition 1.4.3 (isotropic manifold)**.** Let (M,g) be a Riemannian manifold.

- Let $x\in M$. Then (M,g) is *isotropic at x* if the isotropy representation at x acts transitively on the set

  $$\bigl\{v\in \mathrm{T}_x M \bigm| g_x(v\otimes v)=1\bigr\}\subset \mathrm{T}_x M$$

  of unit vectors of $\mathrm{T}_x M$.

- The Riemannian manifold (M,g) is *isotropic* if it is isotropic at every point of M.

Proposition 1.4.4. *Let (M,g) be a Riemannian manifold. If (M,g) is homogeneous and isotropic at a single point, then (M,g) is isotropic.*

Proof. Let $x\in M$ and let (M,g) be isotropic at x. Moreover, let $y\in M$. Then (M,g) is also isotropic at y:

Let $v,w\in \mathrm{T}_y M$ be g_y-unit vectors. We only need to use homogeneity to conjugate the situation from $\mathrm{T}_y M$ to the isotropic situation at x: Because (M,g) is homogeneous, there exists an isometry $f\in\operatorname{Isom}(M,g)$ with

$$f(x)=y.$$

As f is an isometry, $v_x := \mathrm{d}_y(f^{-1})(v)$ and $w_x := \mathrm{d}_y(f^{-1})(w)$ are g_x-unit tangent vectors in $\mathrm{T}_x M$. Moreover, as (M,g) is isotropic at x, there exists an isometry $f_x\in\operatorname{Isom}(M,g)_x$ with

$$\mathrm{d}_x f_x(v_x)=w_x.$$

We now consider the isometry

$$\overline{f} := f\circ f_x\circ f^{-1}\in\operatorname{Isom}(M,g).$$

Then, we obtain

$$\begin{aligned}\overline{f}(y) &= f\circ f_x\circ f^{-1}(y)\\ &= f\circ f_x(x) && (\text{Why?})\\ &= f(x)=y\end{aligned}$$

and

Figure 1.14.: A point reflection at the purple point, schematically

$$
\begin{aligned}
\mathrm{d}_y \overline{f}(v) &= (\mathrm{d}_x f \circ \mathrm{d}_x f_x \circ \mathrm{d}_y(f^{-1}))(v) && \text{(chain rule)} \\
&= (\mathrm{d}_x f \circ \mathrm{d}_x f_x)(v_x) && \text{(by definition of } v_x\text{)} \\
&= \mathrm{d}_x f(w_x) && \text{(by the choice of } f_x\text{)} \\
&= (\mathrm{d}_x f \circ \mathrm{d}_y(f^{-1}))(w) && \text{(by definition of } w_x\text{)} \\
&= \mathrm{d}_y \operatorname{id}_M(w) && \text{(chain rule)} \\
&= w
\end{aligned}
$$

Hence, (M, g) is isotropic at y. □

Study note. Illustrate the proof of Proposition 1.4.4 with suitable pictures!

For the introduction of (locally) symmetric spaces, we will need the notion of a point reflection (Figure 1.14).

Quick check 1.4.5 (point reflection)**.** In terms of linear algebra, how can a point reflection at the origin in Euclidean space be described?

Hint

Definition 1.4.6 ((locally) symmetric spaces)**.** Let (M, g) be a Riemannian manifold.

- Let $x \in M$. A *point reflection* of M at x is an isometry $f \in \operatorname{Isom}(M, g)$ with
$$f(x) = x \quad \text{and} \quad \mathrm{d}_x f = -\operatorname{id}_{\mathrm{T}_x M}.$$
- The Riemannian manifold (M, g) is a *(Riemannian) symmetric space* if M is connected and for every $x \in M$, there exists a point reflection of M at x.

- The Riemannian manifold (M, g) is a *locally symmetric space* if the following holds: For every $x \in M$ there exists an open neighbourhood $U \subset M$ of x such that $(U, g|_U)$ has a point reflection at x.

Proposition 1.4.7. *Let (M, g) be a connected Riemannian manifold. If (M, g) is homogeneous and admits a point reflection at a single point, then (M, g) is a symmetric space.*

Proof. We proceed as in the proof of Proposition 1.4.4: Let $x \in M$ be a point at which (M, g) admits a point reflection r and let $y \in M$.

Because (M, g) is homogeneous, there exists an isometry $f \in \operatorname{Isom}(M, g)$ with $f(x) = y$. Then the conjugate $\overline{r} := f \circ r \circ f^{-1}$ lies in $\operatorname{Isom}(M, g)_y$ (check!) and

$$\begin{aligned}
\mathrm{d}_y \overline{r} &= \mathrm{d}_x f \circ \mathrm{d}_x r \circ \mathrm{d}_y(f^{-1}) && \text{(chain rule)} \\
&= \mathrm{d}_x f \circ (-\operatorname{id}_{\mathrm{T}_x M}) \circ \mathrm{d}_y(f^{-1}) \\
&= -\mathrm{d}_x f \circ \mathrm{d}_y(f^{-1}) && (\text{Why?}) \\
&= -\operatorname{id}_{\mathrm{T}_y M} && \text{(chain rule)}
\end{aligned}$$

Therefore, $\overline{r}$ is a point reflection of (M, g) at y. □

Even if one is not directly interested in symmetry of manifolds, often symmetries can be exploited to simplify arguments or computations.

Outlook 1.4.8 (different levels of symmetry)**.** We do not yet have the means to prove the following facts, but it might be helpful to already keep them in mind:

- Riemannian manifolds can be homogeneous without being isotropic.
- There exist Riemannian manifolds that are isotropic at one point, but that are not isotropic.
- Every symmetric space is homogeneous.
- But not every homogeneous space is symmetric.

The theory of locally symmetric spaces is closely related to the classification of Lie groups [15]: Isometry groups of Riemannian manifolds are Lie groups (e.g., with respect to the compact-open topology); thus, homogeneous spaces are coset spaces of Lie groups.

1.4.2 Euclidean spaces

For the sake of completeness, we quickly summarise what we already know about Euclidean spaces, viewed as Riemannian manifolds (Example 1.3.3, Example 1.3.20): Let $n \in \mathbb{N}$.

- The Euclidean Riemannian metric on $\mathbb{R}^n$ is the Riemannian metric defined by (in the standard coordinates $x^1, \dots, x^n$)

$$\sum_{j=1}^{n} (\mathrm{d}\, x^j)^2 = \sum_{j=1}^{n} (\mathrm{d}\, x^j \otimes \mathrm{d}\, x^j).$$

- All translations and all elements of the orthogonal group $\mathrm{O}(n)$ lie in the Riemannian isometry group of $\mathbb{R}^n$. We will later see that indeed

$$\begin{aligned} \mathbb{R}^n \rtimes \mathrm{O}(n) &\longrightarrow \mathrm{Isom}(\mathbb{R}^n, \text{Euclidean Riemannian metric}) \\ (v, A) &\longmapsto \bigl(x \mapsto A \cdot x + v\bigr) \end{aligned}$$

  is an isomorphism (Theorem 3.4.6).

  In particular, $\mathbb{R}^n$ is homogenous (use translations), isotropic (use rotations or Gram-Schmidt), and symmetric (use classical point reflections).

1.4.3 Spheres

In addition to the standard round sphere, we also introduce (round) spheres of other radii (which slightly generalises Example 1.3.13). Moreover, we have a look at the symmetry properties of spheres.

Definition 1.4.9 (sphere)**.** Let $n \in \mathbb{N}$ and let $R \in \mathbb{R}_{>0}$. We then write

$$\mathbb{S}^n(R) := \bigl\{x \in \mathbb{R}^{n+1} \bigm| \|x\|_2 = R\bigr\} \subset \mathbb{R}^{n+1}$$

for the *sphere of radius R*. We equip this sphere with the first fundamental form (induced from the Euclidean Riemannian metric on $\mathbb{R}^{n+1}$). We denote the corresponding Riemannian manifold usually just by $\mathbb{S}^n(R)$.

Proposition 1.4.10 (isometries of spheres)**.** *Let $n \in \mathbb{N}$ and let $R \in \mathbb{R}_{>0}$. Then*

$$\begin{aligned} \mathrm{O}(n+1) &\longrightarrow \mathrm{Isom}\bigl(\mathbb{S}^n(R)\bigr) \\ A &\longmapsto (x \mapsto A \cdot x) \end{aligned}$$

is a well-defined group homomorphism. In particular, $\mathbb{S}^n(R)$ is homogeneous, isotropic, and symmetric (if $n > 0$).

Proof. A straightforward calculation shows that the specified map is a well-defined group homomorphism (check!); we will later see that this is an isomorphism (Theorem 3.4.7). Therefore, $\mathbb{S}^n(R)$ is

- homogeneous: This is a standard fact from linear algebra; the orthogonal group acts transitively on the unit sphere and whence also on all other spheres centred at the origin.

- isotropic: In view of homogeneity, we only need to check that $\mathbb{S}^n(R)$ is isotropic at the north pole $N := (0, \dots, 0, R)$ (Proposition 1.4.4): The group
$$G := \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; A \in \mathrm{O}(n) \right\} \subset \mathrm{O}(n+1)$$
lies in the stabiliser group $\mathrm{Isom}(\mathbb{S}^n(R))_N$ (check!). Moreover, the isotropy action of G on $\mathrm{T}_N \mathbb{S}^n(R)$ is isomorphic (including the inner product) to the standard action of $\mathrm{O}(n)$ on $\mathbb{R}^n$. The latter action is transitive on the unit sphere; thus also the isotropy action on $\mathrm{T}_N \mathbb{S}^n(R)$ is transitive on the corresponding unit sphere.

- symmetric: In view of homogeneity, we only need to check that the north pole N admits a point reflection. The map
$$\begin{aligned} \mathbb{S}^n(R) &\longrightarrow \mathbb{S}^n(R) \\ x &\longmapsto \boxed{?} \end{aligned}$$
is such a point reflection (check!). Note that $\mathbb{S}^0$ is *not* connected. $\square$

1.4.4 Hyperbolic spaces

Dually to spheres, we can introduce hyperbolic spaces (Figure 1.15). Proceeding literally in this way leads to the hyperboloid model of hyperbolic spaces. Depending on the type of application in mind, other models can also be convenient (even though their Riemannian metrics might look non-intuitive at first). Therefore, we will also introduce the Poincaré disk model and the Poincaré halfspace model.

Moreover, as in the Euclidean and the spherical case, we will establish that hyperbolic spaces are homogeneous, isotropic, and symmetric.

Proposition and Definition 1.4.11 (hyperbolic space). *Let $n \in \mathbb{N}$. Then the following Riemannian manifolds of dimension n are isometric; we refer to them as* hyperbolic n-space of radius R *and usually abbreviate this by $\mathbb{H}^n(R)$.*

1. Hyperboloid model. *As underlying set we take*
$$\mathbb{H}^n(R) := \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; t > 0 \text{ and } x_1^2 + \dots + x_n^2 - t^2 = -R^2 \right\} \subset \mathbb{R}^{n+1}$$
with the covariant 2-tensor $g^{\mathbb{H}}$ induced by the following 2-tensor on $\mathbb{R}^{n+1}$ (in coordinates $x^1, \dots, x^n, t$):
$$(\mathrm{d}\, x^1)^2 + \dots + (\mathrm{d}\, x^n)^2 - (\mathrm{d}\, t)^2.$$
More explicitly, this bilinear form on $\mathbb{R}^{n+1}$ is given by

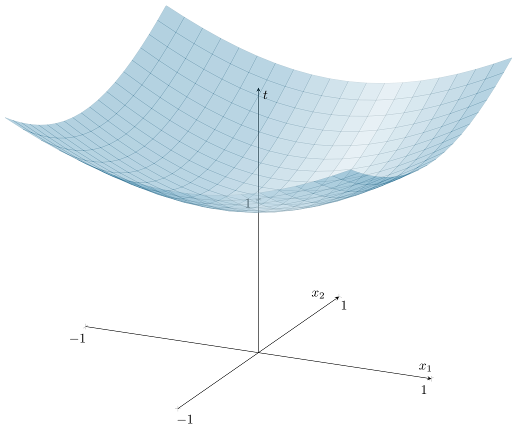


Figure 1.15.: The two-dimensional hyperboloid (of radius 1); however, one has to keep in mind that the Riemannian metric on this hyperboloid is *not* the one induced by the Riemannian Euclidean metric in the ambient Euclidean space!

$$\begin{aligned} \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} &\longrightarrow \mathbb{R} \\ \bigl((x,t),(x',t')\bigr) &\longmapsto \sum_{j=1}^{n} x_j \cdot x'_j - t \cdot t'. \end{aligned}$$

2. Poincaré disk model. *As underlying set we take the Euclidean open R-ball*

$$\mathbb{B}^n(R) := \bigl\{x \in \mathbb{R}^n \bigm| \|x\|_2 < R\bigr\} \subset \mathbb{R}^n.$$

On this open subset of $\mathbb{R}^n$, we consider the Riemannian metric given by (in the coordinates $x^1, \dots, x^n$)

$$\frac{4 \cdot R^4}{(R^2 - \|x\|_2{}^2)^2} \cdot \bigl((\mathrm{d}\, x^1)^2 + \dots + (\mathrm{d}\, x^n)^2\bigr).$$

3. Poincaré halfspace model. *As underlying set we take the upper halfspace*

$$\mathbb{U}^n(R) := \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \subset \mathbb{R}^n.$$

On this open subset of $\mathbb{R}^n$, we consider the Riemannian metric given by (in the coordinates $x^1, \dots, x^{n-1}, y$)

$$\frac{R^2}{y^2} \cdot \bigl((\mathrm{d}\, x^1)^2 + \dots + (\mathrm{d}\, x^{n-1})^2 + (\mathrm{d}\, y)^2\bigr).$$

Proof. Clearly, $\mathbb{B}^n(R)$ and $\mathbb{U}^n(R)$ are smooth manifolds of dimension n (as open subsets of Euclidean spaces). Why/how is $\mathbb{H}^n(R)$ a smooth manifold of dimension n ? The definition of $\mathbb{H}^n(R)$ cries for an application of the What? Dually to the sphere case (Example 1.1.33), we consider the function

$$\begin{aligned} f\colon \mathbb{R}^n \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x,t) &\longmapsto \sum_{j=1}^n x_j^2 - t^2 + R^2. \end{aligned}$$

Then 0 is a regular value of f (check!) and thus $\mathbb{H}^n(R) = f^{-1}(\{0\}) \cap (\mathbb{R}^n \times \mathbb{R}_{>0})$ is a smooth submanifold of $\mathbb{R}^{n+1}$ of dimension n.

Furthermore, $g^{\mathbb{B}}$ and $g^{\mathbb{U}}$ are Riemannian metrics on $\mathbb{B}^n(R)$ and $\mathbb{U}^n(R)$, respectively (Proposition 1.3.5). But it should be noted that we still need to show that $g^{\mathbb{H}}$ indeed is a Riemannian metric on $\mathbb{H}^n(R)$. What's the problem? We will integrate this step into the proof that the hyperboloid model and the Poincaré disk model are isometric: We construct a diffeomorphism $s\colon \mathbb{H}^n(R) \longrightarrow \mathbb{B}^n(R)$ with $s^* g^{\mathbb{B}} = g^{\mathbb{H}}$. In particular, this shows that $g^{\mathbb{H}}$ is a Riemannian metric on $\mathbb{H}^n(R)$ (Proposition 1.3.4).

Comparing hyperboloid and Poincaré disk model. We transform the hyperboloid model into the Poincaré disk model via the *hyperbolic stereographic projection*: Geometrically, this map is defined as follows: For $x \in \mathbb{H}^n(R)$, we consider the line segment from x to the "south pole" $(0, \dots, -R)$. Then the hyperbolic stereographic projection is the intersection of this line segment with $\mathbb{B}^n(R) \times \{0\} \subset \mathbb{R}^{n+1}$. As usual, we will not work with this geometric definition, but with its analytic representation: Let

$$\begin{aligned} s\colon \mathbb{H}^n(R) &\longrightarrow \mathbb{B}^n(R) \\ (x,t) &\longmapsto \frac{R \cdot x}{R+t}. \end{aligned}$$

We now show that s is a diffeomorphism and that $s^* g^{\mathbb{B}} = g^{\mathbb{H}}$:

- Why is s a diffeomorphism? First of all, we compute that the image of s indeed lies in $\mathbb{B}^n(R)$: Let $(x,t) \in \mathbb{H}^n(R)$. Then

$$\begin{aligned} \|s(x,t)\|_2{}^2 &= \frac{R^2 \cdot \|x\|_2{}^2}{(R+t)^2} \\ &= \frac{R^2 \cdot (t^2 - R^2)}{(R+t)^2} && (\text{Why?}) \\ &= \frac{R^2 \cdot (t-R)}{t+R} \\ &< R^2, && (\text{as } x \in \mathbb{H}^n(R) \text{ implies } t > R) \end{aligned}$$

which means that $s(x,t) \in \mathbb{B}^n(R)$.

A straightforward computation shows that s is smooth (check!). Moreover, the map

$$\begin{aligned} \mathbb{B}^n(R) &\longrightarrow \mathbb{H}^n(R) \\ u &\longmapsto \Bigl(\frac{2 \cdot R^2 \cdot u}{R^2 - \|u\|_2{}^2}, R \cdot \frac{R^2 + \|u\|_2{}^2}{R^2 - \|u\|_2{}^2}\Bigr) \end{aligned}$$

is smooth and inverse to s (check!). So, s is a diffeomorphism.

- We have $s^* g^{\mathbb{B}} = g^{\mathbb{H}}$: In view of polarisation, it suffices to show that

$$(s^* g^{\mathbb{B}})_{(x,t)}(v \otimes v) = g^{\mathbb{H}}_{(x,t)}(v \otimes v)$$

holds for all $(x,t) \in \mathbb{H}^n(R)$ and all $v \in \mathrm{T}_{(x,t)} \mathbb{H}^n(R)$ (check!); this reduction is not really necessary, but it reduces the notational overhead a bit. Let $v \in \mathrm{T}_{(x,t)} \mathbb{H}^n(R) \subset \mathbb{R}^{n+1}$, written in components as $v = (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}$.

The definition of $s^* g^{\mathbb{B}}$ involves $\mathrm{d}\, s$, which we compute first: We have

$$\mathrm{d}_{(x,t)}\, s(\xi,\tau) = \boxed{?} \qquad .$$

Therefore, we obtain

$$\begin{aligned} (s^* g^{\mathbb{B}})_{(x,t)}(v \otimes v) &= g^{\mathbb{B}}_{s(x,t)}(\mathrm{d}_{(x,t)}\, s(\xi,\tau) \otimes \mathrm{d}_{(x,t)}\, s(\xi,\tau)) \\ &= \frac{4 \cdot R^4}{(R^2 - \|s(x,t)\|_2{}^2)^2} \cdot \Bigl\| \frac{R}{R+t} \cdot \xi - \frac{R}{(R+t)^2} \cdot \tau \cdot x \Bigr\|_2^2 \\ &= \underbrace{\frac{4 \cdot R^4}{(R^2 - \|s(x,t)\|_2{}^2)^2} \cdot \frac{R^2}{(R+t)^2}}_{=:①} \cdot \underbrace{\Bigl\| \xi - \frac{\tau}{R+t} \cdot x \Bigr\|_2^2}_{=:②}. \end{aligned}$$

We simplify these terms separately:

① Using the computation $\|s(x,t)\|_2{}^2 = R^2 \cdot (t-R)/(t+R)$ from above, we find

$$\begin{aligned} ① &= \frac{4 \cdot R^4}{(R^2 - \frac{R^2 \cdot (t-R)}{(t+R)^2})^2} \cdot \frac{R^2}{(R+t)^2} = \frac{4 \cdot R^4}{(R^2 \cdot (t+R) - R^2 \cdot (t-R))^2} \cdot R^2 \\ &= 4 \cdot \frac{1}{(t+R-(t-R))^2} \cdot R^2 = \frac{4 \cdot R^2}{(2 \cdot R)^2} \\ &= 1. \end{aligned}$$

② Because $(\xi, \tau) \in \mathrm{T}_{(x,t)} \mathbb{H}^n(R)$ (viewed as submanifold tangent space), differentiating the defining equation of $\mathbb{H}^n(R)$ shows that

$$\langle x, \xi\rangle_2 = t \cdot \tau.$$

Therefore, we obtain

$$\begin{aligned}
② &= \langle \xi, \xi\rangle_2 - \frac{2\cdot\tau}{R+t} \cdot \langle x, \xi\rangle_2 + \frac{\tau^2}{(R+t)^2} \cdot \langle x, x\rangle_2 \\
&= \langle \xi, \xi\rangle_2 - \frac{2\cdot\tau}{R+t} \cdot t \cdot \tau + \frac{\tau^2}{(R+t)^2} \cdot \langle x, x\rangle_2 && \text{(because } (\xi,\tau) \in \mathrm{T}_{(x,t)} \mathbb{H}^n(R)\text{)} \\
&= \langle \xi, \xi\rangle_2 - \frac{2\cdot t\cdot\tau^2}{R+t} + \frac{\tau^2 \cdot (t^2 - R^2)}{(R+t)^2} && \text{(because } (x,t) \in \mathbb{H}^n(R)\text{)} \\
&= \langle \xi, \xi\rangle_2 - \tau^2 && \text{(elementary computation)} \\
&= g^{\mathbb{H}}_{(x,t)}\bigl((\xi,\tau) \otimes (\xi,\tau)\bigr) = g^{\mathbb{H}}_{(x,t)}(v \otimes v)
\end{aligned}$$

Putting ① and ② together proves that $s^* g^{\mathbb{B}} = g^{\mathbb{H}}$. In particular, $g^{\mathbb{H}}$ is a Riemannian metric on $\mathbb{H}^n(R)$.

Comparing Poincaré disk and halfspace model. Similarly, we transform the Poincaré halfspace model into the Poincaré disk model via the *Cayley transform*: The Cayley transform

$$\begin{aligned}
c\colon \mathbb{U}^n(R) &\longrightarrow \mathbb{B}^n(R) \\
\mathbb{R}^{n-1} \times \mathbb{R}_{>0} \ni (x,y) &\longmapsto \frac{1}{\|x\|_2{}^2 + (y+R)^2} \bigl(2\cdot R^2 \cdot x, R \cdot (\|x\|_2{}^2 + \|y\|_2{}^2 - R^2)\bigr)
\end{aligned}$$

is well-defined and smooth (check!) and has the inverse map

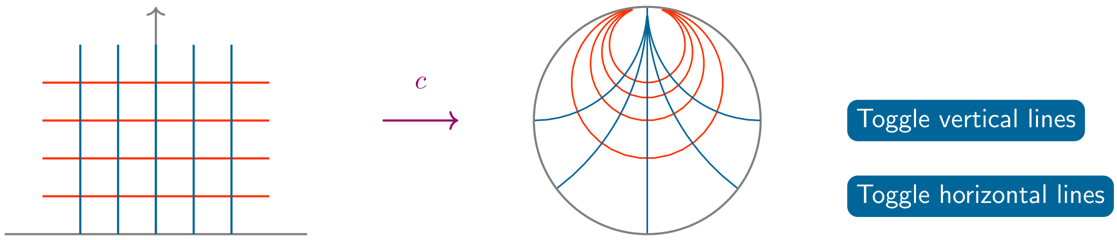
$$\begin{aligned}
\mathbb{B}^n(R) &\longrightarrow \mathbb{U}^n(R) \\
\mathbb{R}^{n-1} \times \mathbb{R} \supset \mathbb{B}^n(R) \ni (u,v) &\longmapsto \frac{1}{\|u\|_2{}^2 + (v-R)^2} \cdot \bigl(2\cdot R^2 \cdot u, R \cdot (R^2 - \|u\|_2{}^2 - v^2\bigr)
\end{aligned}$$

(check!), which is also smooth (check!). The geometric effect of the Cayley transform is illustrated in Figure 1.16.

A computation now shows that $c^* g^{\mathbb{B}} = g^{\mathbb{U}}$ (Exercise; this requires some computational stamina ...). □

Study note (on the proof of Proposition 1.4.11).

- Check that the defining formula for s indeed has the claimed geometric meaning.
- Redo the proof of Proposition 1.4.11, using the "$\mathrm{d}\,x$"-notation for the computations.
- You might know the Cayley transform $z \mapsto i \cdot (z - i)/(z + i)$ from complex analysis. Compare this to the map c in the case $n = 2$!



Poincaré halfspace Poincaré disk

Figure 1.16.: The Cayley transform, schematically

Another popular model of hyperbolic space is the Beltrami-Klein model [18, 30], which is also based on an open ball.

Proposition 1.4.12 (symmetry of hyperbolic space). *Let $n \in \mathbb{N}$. Then hyperbolic n-space $\mathbb{H}^n(R)$ is homogeneous, isotropic, and symmetric.*

Proof. We first show that $\mathbb{H}^n(R)$ is *homogeneous*. There are many ways of doing this. For instance, we can use the halfspace model (by Proposition 1.4.11, the models of hyperbolic space are isometric): We consider the following two types of isometries:

- For $a \in \mathbb{R}^{n-1}$, the horizontal translation map
$$\begin{aligned} \mathbb{U}^n(R) &\longrightarrow \mathbb{U}^n(R) \\ \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \ni (x, y) &\longmapsto (x + a, y) \end{aligned}$$
is an isometry with respect to $g^{\mathbb{U}}$ (check!).

- For $\lambda \in \mathbb{R}_{>0}$, the scaling map
$$\begin{aligned} f \colon \mathbb{U}^n(R) &\longrightarrow \mathbb{U}^n(R) \\ z &\longmapsto \lambda \cdot z \end{aligned}$$
is an isometry with respect to $g^{\mathbb{U}}$: Clearly, f is a diffeomorphism and for all $z \in \mathbb{U}^n(R)$, we have (via the canonical identification of the tangent spaces of $\mathbb{U}^n(R)$ with $\mathbb{R}^n$)
$$\forall_{v \in \mathrm{T}_z \mathbb{U}^n(R)} \quad \mathrm{d}_z f(v) = \lambda \cdot v.$$
Therefore, for all $(x, y) \in \mathbb{U}^n(R)$ and all $v, w \in \mathrm{T}_{(x,y)} \mathbb{U}^n(R)$, we obtain

$$
\begin{aligned}
f^* g^{\mathbb{U}}_{(x,y)}(v \otimes w) &= g^{\mathbb{U}}_{f(x,y)}\bigl(\mathrm{d}_{(x,y)} f(v) \otimes \mathrm{d}_{(x,y)} f(w)\bigr) \\
&= g^{\mathbb{U}}_{(\lambda \cdot x, \lambda \cdot y)}(\lambda \cdot v \otimes \lambda \cdot w) \\
&= \lambda^2 \cdot \frac{R^2}{\lambda^2 \cdot y^2} \cdot \langle v, w \rangle_2 = \frac{R^2}{y^2} \cdot \langle v, w \rangle_2 \\
&= g^{\mathbb{U}}_{(x,y)}(v \otimes w).
\end{aligned}
$$

Combining these two types of isometries, $(0, \dots, 0, 1) \in \mathbb{U}^n(R)$ can be transported via isometries to any other point in $\mathbb{U}^n(R)$ (because

). Therefore, detours through $(0, \dots, 1)$ show that $\operatorname{Isom}(\mathbb{U}^n(R), g^{\mathbb{U}})$ acts transitively on $\mathbb{U}^n(R)$.

It remains to show that $\mathbb{H}^n(R)$ is *isotropic* and *symmetric*. For these aspects, we will use the hyperboloid model to exploit the "rotational symmetry" of the hyperboloid. In view of homogeneity, it suffices to show that $\mathbb{H}^n(R)$ is isotropic at the north pole $N := (0, \dots, 0, R)$ and that $\mathbb{H}^n(R)$ admits a point reflection at N.

The map

$$
\begin{aligned}
\varphi \colon \mathrm{O}(n) &\longrightarrow \operatorname{Isom}(\mathbb{H}^n(R))_N \\
A &\longmapsto \bigl((x, y) \mapsto (A \cdot x, y)\bigr)
\end{aligned}
$$

is a well-defined group homomorphism (check!) and for each $A \in \mathrm{O}(n)$, we have that $\mathrm{d}_N(\varphi(A)) \colon \mathrm{T}_N \mathbb{H}^n(R) \longrightarrow \mathrm{T}_N \mathbb{H}^n(R)$ is represented by A with respect to the standard basis of the submanifold tangent space $\mathrm{T}_N \mathbb{H}^n(R) = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. Moreover, under this identification, $g^{\mathbb{H}}_N$ coincides with the Euclidean inner product on $\mathbb{R}^n$.

Therefore, we can argue as in the case of $\mathbb{S}^n(R)$ (Proposition 1.4.10) to see that $\mathbb{H}^n(R)$ is isotropic at N and that $\mathbb{H}^n(R)$ admits a point reflection at N (check!). $\square$

It is common to abbreviate $\mathbb{H}^n(1)$ by $\mathbb{H}^n$.

Outlook 1.4.13. Hyperbolic geometry has a rich and fascinating history (starting with the parallel postulate by Euclid). We will return to this later (Remark **??**). By now, hyperbolic geometry found its way into art and popular culture:

- Escher met Coxeter and was inspired to create hyperbolic art (e.g., *Cirkellimiet I, IV* [9]).
- There exist instructions for hyperbolic needlework [4].
- There are computer games based on regular tilings of the hyperbolic plane, e.g., HyperRogue [32].

1.4.5 Group actions

For objects with a high degree of symmetry, it can be useful to divide out (some of) these symmetries and to consider the corresponding quotients. In order to obtain quotients with good topological and geometric properties, we will have to impose restrictions on the group action. The simplest version is to require both freeness and properness.

We will first introduce the notion of proper actions and discuss how properness in combination with freeness leads to good quotients. We then give some classical examples.

Throughout, we will restrict ourselves to actions by discrete groups (so that we do not need to introduce technicalities on topological groups or Lie groups).

Definition 1.4.14 (proper action)**.** Let $\Gamma \curvearrowright X$ be a continuous action of a (discrete) group Γ on a topological space X. This action is *proper* if for each compact set $K \subset X$, the "return" set

$$\{\gamma \in \Gamma \mid K \cap \gamma \cdot K \neq \emptyset\}$$

is finite.

Quick check 1.4.15. Are the following actions proper?

1. The translation action $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$.
   Yes No
2. The translation action $\mathbb{Q}^2 \curvearrowright \mathbb{R}^2$.
   Yes No
3. The action $\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{R}^2$ by matrix multiplication.
   Yes No
4. Rotation actions $\mathbb{Z} \curvearrowright \mathbb{S}^1$.
   Yes No

Caveat 1.4.16 (free vs. proper)**.** In general,

- free actions are *not* proper (e.g., irrational rotation actions on the circle), and
- proper actions are *not* free (e.g., trivial actions by non-trivial finite groups on non-empty spaces).

Remark 1.4.17 (free + proper)**.** We will use the following key fact from topology: Let $\Gamma \curvearrowright X$ be a free and proper action on a topological manifold X. Then this action is a so-called properly discontinuous action (check!): For each $x \in X$, there exists an open neighbourhood $U \subset X$ with

$$\forall_{\gamma \in \Gamma \setminus \{e\}} \quad U \cap \gamma \cdot U = \emptyset.$$

Hence, the quotient map $p\colon X \longrightarrow \Gamma \setminus X$ (where $\Gamma \setminus X$ carries the quotient topology) is a *covering map* (Proposition AT.2.3.7), i.e., a locally trivial fibre bundle with discrete fibre Γ: For each $y \in Y := \Gamma \setminus X$, there exists an open neighbourhood $U \subset Y$ and a homeomorphism $f\colon p^{-1}(U) \longrightarrow U \times \Gamma$ with

$$p|_{p^{-1}(U)} = (\text{projection } q \text{ onto the first factor}) \circ f$$

$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "f"] \arrow[d, "p|"'] & U \times \Gamma \arrow[d, "q"] \\
U \arrow[r, equal] & U
\end{tikzcd}$$

In particular, p is open and a local homeomorphism (check!).

Furthermore, we can choose U and f in such a way that $f(\gamma \cdot x) = \gamma \cdot f(x)$ holds for all $x \in p^{-1}(U)$ and all $\gamma \in \Gamma$, where Γ acts on the second factor of $U \times \Gamma$ by translation from the left. In this situation, the sets of the form $f^{-1}(U \times \{\gamma\})$ with $\gamma \in \Gamma$ are called *sheets* of p over U.

Conversely, by covering theory, every regular covering of a topological manifold (with countable fibre) is isomorphic to the quotient map of a suitable free and proper action on a topological manifold.

Study note. Compare the definition of covering maps with the definition of vector bundles!

Proposition 1.4.18 (smooth structures on quotient manifolds)**.** *Let $\Gamma \curvearrowright M$ be a smooth action of a (discrete) group Γ on a smooth manifold M that is free and proper. Then, the topological quotient space $\Gamma \setminus M$ carries a unique smooth structure that turns the canonical projection map $M \longrightarrow \Gamma \setminus M$ into a local diffeomorphism.*

Proof. Let $p\colon M \longrightarrow N := \Gamma \setminus M$ denote the canonical projection map. Because the action is free and proper, p is a covering map (Remark 1.4.17).

Using the local triviality of p, we can construct a smooth atlas on the quotient N: Let $y \in N$ and let $f_y\colon p^{-1}(U_y) \longrightarrow U_y \times \Gamma$ be a local trivialisation of p around y.

Let $x \in p^{-1}(U_y)$ and let $V \subset p^{-1}(U_y)$ be the sheet of $p^{-1}(U_y)$ that contains x. Then V is open (check!). Let $\varphi\colon W \longrightarrow W'$ be a smooth chart around x with $W \subset V$. Then $W_y := p(W) \subset N$ is open, $p|_W\colon W \longrightarrow W_y$ is a homeomorphism, and we take $\varphi_y := \varphi \circ p|_W^{-1}\colon W_y \longrightarrow W'$ as a chart for N.

The family $(\varphi_y)_{y\in N}$ forms a smooth atlas for N (check! How does the change of charts relate to the change of charts on M?).

By construction, $p\colon M \longrightarrow N$ is a local diffeomorphism with respect to this smooth structure.

Moreover, requiring p to be a local diffeomorphism clearly uniquely determines the smooth structure on N (check!). □

Caveat 1.4.19. In general, neither freeness nor properness are sufficient to guarantee that the quotient is a manifold:

- Irrational rotations on the circle are free, but the quotients are not even Hausdorff (check!).
- The action of $\mathbb{Z}/2$ on $\mathbb{S}^1$ by reflection at the first coordinate axis is proper (as $\mathbb{Z}/2$ is finite), but the quotient is homeomorphic to a closed interval (whose endpoints do not admit Euclidean neighbourhoods).

Proposition 1.4.20 (Riemannian metrics on quotient manifolds)**.** *Let (M, g) be a Riemannian manifold and let $\Gamma \curvearrowright M$ be an isometric action of a (discrete) group Γ on (M, g) that is free and proper. Then, the smooth quotient manifold $\Gamma \setminus M$ carries a unique Riemannian metric that turns the canonical projection map $M \longrightarrow \Gamma \setminus M$ into a local isometry.*

Proof. Let $p\colon M \longrightarrow N := \Gamma \setminus M$ denote the canonical projection. Because the action is free and proper, p is a covering map (Remark 1.4.17) and, by construction of the smooth structure on N (Proposition 1.4.18), p is a local diffeomorphism.

We now construct the desired Riemannian metric on N locally: Let $U \subset N$ be an open subset over which p is trivial (in the sense of covering maps). Then, there exists a sheet $W \subset p^{-1}(U)$ such that $p|_W\colon W \longrightarrow U$ is a diffeomorphism. In particular,

$$h_U := (p|_W{}^{-1})^* g|_W$$

is a Riemannian metric on U (Proposition 1.3.4).

Moreover, this construction of h_U is independent of the choice of W: Every other sheet V of p over U differs from W by the action of an element in Γ: There exists a $\gamma \in \Gamma$ with $p|_V = p|_W \circ (\gamma \cdot \ \cdot\ )$. As Γ acts by isometries on (M, g), the resulting values for h_U coincide (check!).

Therefore, if $(U_i)_{i\in I}$ is a cover of N by open sets over which p trivalises, the local Riemannian metrics $(h_{U_i})_{i\in I}$ satisfy

$$\forall_{i,j\in I} \quad h_{U_i}|_{U_i\cap U_j} = h_{U_j}|_{U_i\cap U_j}$$

and thus glue to a Riemannian metric on N (check!). □

Remark 1.4.21 (from the quotient to the total space)**.** Let M be a topological space and let $\Gamma \curvearrowright M$ be a free and proper action by a countable group whose quotient space $\Gamma \curvearrowright M$ carries the structure of a smooth manifold. Moreover, let $p\colon M \longrightarrow \Gamma \setminus M$ denote the canonical projection.

- Then, p is a local homeomorphism (Remark 1.4.17).
- In particular: The smooth structure on $\Gamma \setminus M$ can be pulled back via p to a smooth structure on M (check!) and the given action $\Gamma \curvearrowright M$ on M is an action by diffeomorphisms. With this smooth structure, p is a local diffeomorphism.
- If g is a Riemannian metric on $\Gamma \setminus M$, then p^*g is a Riemannian metric on M (Proposition 1.3.4) and the given action $\Gamma \curvearrowright M$ on M is isometric with respect to p^*g (check!). With this Riemannian metric, p is a local isometry.

Example 1.4.22 (real projective spaces). Let $n \in \mathbb{N}$. Then the antipodal action

$$\begin{aligned} \mathbb{Z}/2 \times \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ \bigl([m], x\bigr) &\longmapsto (-1)^m \cdot x \end{aligned}$$

on the sphere $\mathbb{S}^n$ is free (thus proper, as $\mathbb{Z}/2$ is a finite group) and isometric. The quotient

$$\mathbb{R}P^n := (\mathbb{Z}/2) \setminus \mathbb{S}^n$$

with the induced smooth structure is the *n-dimensional real projective space*. Because the action is isometric, the round Riemannian metric on $\mathbb{S}^n$ induces a Riemannian metric on $\mathbb{R}P^n$ (Proposition 1.4.20). We will usually equip $\mathbb{R}P^n$ with this Riemannian metric.

Even though $\mathbb{S}^2$ is a smooth submanifold in $\mathbb{R}^3$, one can prove that the quotient $\mathbb{R}P^2$ is *not* diffeomorphic to a smooth submanifold of $\mathbb{R}^3$ [21].

Outlook 1.4.23 (complex projective space). Generalising the construction of quotients from actions of discrete groups to suitable actions by Lie groups, one can also define for $n \in \mathbb{N}$ the *n-dimensional complex projective space*

$$\mathbb{C}P^n := \mathbb{S}^1 \setminus \mathbb{S}^{2n+1},$$

where $\mathbb{S}^1 \subset \mathbb{C}^\times$ acts on $\mathbb{S}^{2n+1} \subset \mathbb{C}^n$ by complex multiplication. The induced Riemannian metric on $\mathbb{C}P^n$ is also called *Fubini-Study metric*. It should be noted that "n-dimensional" refers to the $\mathbb{C}$-dimension; thus, as a real manifold, we have $\dim \mathbb{C}P^n = 2 \cdot n$.

Example 1.4.24 (tori). Let $n \in \mathbb{N}$. Then the translation action $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ is free, proper, and isometric with respect to the Euclidean Riemannian metric on $\mathbb{R}^n$. The quotient manifold $\mathbb{Z}^n \setminus \mathbb{R}^n$ is diffeomorphic to the n-torus $(\mathbb{S}^1)^{\times n}$. Moreover, $\mathbb{Z}^n \setminus \mathbb{R}^n$ with the Riemannian metric induced from the Euclidean Riemannian metric on $\mathbb{R}^n$ (Proposition 1.4.20) is isometric to $(\mathbb{S}^1(1/(2\pi))^{\times n}$ with the product of the round metric on $\mathbb{S}^1(1/(2\pi))$ (check! use Example 1.3.20).

Example 1.4.25 (Möbius strip)**.** The Möbius strip can be modelled as the quotient manifold of the action

$$\begin{aligned} \mathbb{Z} \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \big(m, (x,y)\big) &\longmapsto \boxed{?} \end{aligned}$$

.

This construction is diffeomorphic to the vector bundle construction (Example 1.2.11).

Remark 1.4.26 (locally symmetric spaces from group actions)**.** Let (M, g) be a symmetric space and let $\Gamma \curvearrowright (M, g)$ be an isometric free and proper action by a (discrete) group Γ. Then, $\Gamma \setminus M$ with the induced Riemannian metric is a locally symmetric space (Exercise; the most elementary proof needs the Riemannian distance function, though).

In fact, also the converse holds: Every connected locally symmetric space is isometric to a quotient of a symmetric space by an isometric free and proper action of a discrete group (this is much harder to prove) [15].

1.5 Towards Riemannian geometry

Finally, we explain how to extract basic geometric data and invariants from Riemannian metrics – most notably an actual metric. This works as follows: Let (M, g) be a Riemannian manifold.

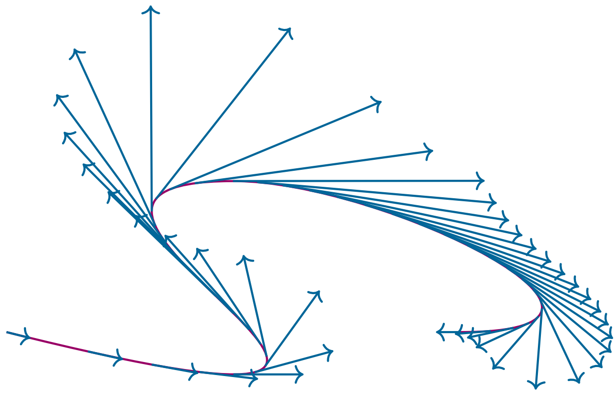
- The Riemannian metric g allows to measure the length of tangent vectors of M.
- As in physics, the length of a curve on M is nothing but the integral of the speed (length of the velocity vector!) along the curve.
- The distance between two points on M is the infimum over the lengths of all curves connecting these points.

We will now work this out in more detail.

Moreover, we will also briefly explain how to define the volume of M in the presence of a Riemannian metric.

1.5.1 Lengths of curves

The length of a curve on a Riemannian manifold is the integral of the speed (measured via the Riemannian metric) along the curve (Figure 1.17). As we will also want to consider concatenations of curves, we will slightly generalise the setup from smooth curves to piecewise smooth curves.



Toggle velocity vectors

Figure 1.17.: Velocity vectors along a curve, schematically; the speed is then measured as the length of the velocity vectors with respect to the Riemannian metric.

Definition 1.5.1 (piecewise smooth/regular curve)**.** Let M be a smooth manifold and let $a, b \in \mathbb{R}$ with $a < b$.

- A *smooth curve* $[a, b] \longrightarrow M$ is a map $[a, b] \longrightarrow M$ that admits a smooth extension to an open interval containing $[a, b]$.

  A smooth curve $\gamma\colon [a, b] \longrightarrow M$ is *regular* if $\mathrm{d}_t\, \gamma \neq 0$ for all $t \in [a, b]$.

- A *piecewise smooth curve* $[a, b] \longrightarrow M$ is a map $\gamma\colon [a, b] \longrightarrow M$ for which there exists a $k \in \mathbb{N}$ and $a = a_0 < \cdots < a_k = b$ with the property that $\gamma|_{[a_0,a_1]}, \dots, \gamma|_{[a_{k-1},a_k]}$ are smooth curves.

  The curve γ is *piecewise regular*, if each of the restrictions $\gamma|_{[a_j,a_{j+1}]}$ with $j \in \{0, \dots, k-1\}$ is regular.

Definition 1.5.2 (length of a curve)**.** Let (M, g) be a Riemannian manifold and let $\gamma\colon [a, b] \longrightarrow M$ be a piecewise regular curve on M. Then the *length of γ with respect to g* is defined by

$$L_g(\gamma) := \int_a^b \big\|\dot\gamma(t)\big\|_{g_{\gamma(t)}}\, dt.$$

Here,

- $\dot\gamma(t)$ denotes the tangent vector represented by γ at t (i.e., the tangent vector represented by $s \mapsto \gamma(t + s)$ at 0), and

- $\|\cdot\|_{g_x}$ denotes for $x \in M$ the norm on $\mathrm{T}_x M$ induced by the inner product g_x.

Remark 1.5.3 (on the definition of length)**.** Let (M, g) be a Riemannian manifold and let $\gamma\colon [a, b] \longrightarrow M$ be a piecewise regular curve in M. Then γ is smooth, except for finitely many places; moreover the differential of γ is smooth, except for the same finitely many exceptions. Therefore, the integral $\int_a^b \|\dot\gamma(t)\|_{g_{\gamma(t)}}\,dt$ over the lengths of the tangent vectors is well-defined.

Example 1.5.4 (length of curves)**.** We compute the length of some curves in model spaces:

- Let $n \in \mathbb{N}$, let $a, b \in \mathbb{R}^n$, and let

$$\begin{aligned} \gamma\colon [0,1] &\longrightarrow \mathbb{R}^n \\ t &\longmapsto a + t \cdot (b - a) \end{aligned}$$

be the straight line segment from a to b. Then, for the Euclidean Riemannian metric g on $\mathbb{R}^n$, we obtain

$$L_g(\gamma) = \boxed{?} \qquad .$$

- Let g denote the round metric on $\mathbb{S}^1$ and let

$$\begin{aligned} \gamma\colon [0,1] &\longrightarrow \mathbb{S}^1 \\ t &\longmapsto \bigl(\cos(\pi \cdot t), \sin(\pi \cdot t)\bigr) \end{aligned}$$

be a parametrisation of the semi-circle. Then, we obtain

$$L_g(\gamma) = \boxed{?}$$

- Let $y \in \mathbb{R}_{>0}$. Then the vertical segment

$$\begin{aligned} \gamma_y\colon [0,1] &\longrightarrow \mathbb{U}^2(1) \\ t &\longmapsto (0, t + y) \end{aligned}$$

satisfies

$$L_{g^{\mathbb{U}}}(\gamma_y) = \boxed{?}$$

In contrast, the horizontal segment

$$\begin{aligned} \eta_y\colon [0,1] &\longrightarrow \mathbb{U}^2(1) \\ t &\longmapsto (t, y) \end{aligned}$$

satisfies

$$L_{g^{\mathbb{U}}}(\eta_y) = \boxed{?}$$

Proposition 1.5.5 (isometry invariance of lengths). *Let $f\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be a local isometry and let γ be a piecewise regular curve in M_1. Then*

$$L_{g_2}(f \circ \gamma) = L_{g_1}(\gamma).$$

Proof. This is a straightforward computation: Let γ be defined on $[a,b]$ and let $\eta := f \circ \gamma$ (which is a piecewise regular curve on M_2; check!). Then, by definition of the length of curves, we have

$$\begin{aligned} L_{g_2}(f \circ \gamma) &= \int_a^b \big\|\dot\eta(t)\big\|_{(g_2)_{\eta(t)}} \, dt \\ &= \int_a^b \big\|\mathrm{d}_{\gamma(t)} f(\dot\gamma(t))\big\|_{(g_2)_{\eta(t)}} \, dt \quad \text{(chain rule, with finitely many exceptions)} \\ &= \int_a^b \big\|\dot\gamma(t)\big\|_{(g_1)_{\gamma(t)}} \, dt \qquad (f \text{ is a local isometry}) \\ &= L_{g_1}(\gamma), \end{aligned}$$

as claimed. □

Remark 1.5.6 (parametrisation by arc-length). Let us recall the following facts on regular curves from multi-variable analysis, which directly translate into the manifold setting (check!): Let (M, g) be a Riemannian manifold and let $\gamma\colon [a,b] \longrightarrow M$ be a regular curve.

- We say that γ is *parametrised by arc-length* if

$$\forall_{t \in [a,b]} \quad \big\|\dot\gamma(t)\big\|_{g_{\gamma(t)}} = 1.$$

- The regular curve γ admits exactly one orientation preserving smooth reparametrisation that is parametrised by arc-length [18, Proposition 2.59].

Analogous results also hold for piecewise regular curves.

1.5.2 The Riemannian distance function

We now define the distance between two points on a Riemannian manifold as the infimum over the lengths of all piecewise regular curves connecting these points (Figure 1.18).

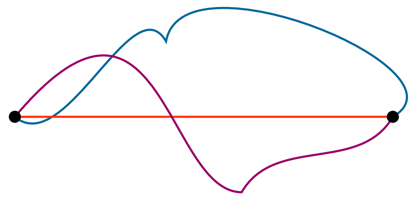


Figure 1.18.: Piecewise regular curves between two points, schematically

Clearly, this notion is invariant under Riemannian isometries and thus provides a good geometric setup on Riemannian manifolds.

In order to prove that the Riemannian distance function indeed is a metric on the underlying manifold that induces the original topology, we need some preparations:

- We first show that in the Euclidean case we get back the usual Euclidean metric.
- We then locally compare general Riemannian metrics with the Euclidean case.

Definition 1.5.7 (Riemannian distance)**.** Let (M, g) be a Riemannian manifold. Then the *Riemannian distance function associated with* g is defined by

$$\begin{aligned} d_g \colon M \times M &\longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\} \\ (x, y) &\longmapsto \inf \bigl\{ L_g(\gamma) \bigm| \gamma \text{ is a piecewise regular curve in } M \text{ from } x \text{ to } y. \bigr\} \end{aligned}$$

Study note. Compare this definition of distance with the notion of distance in route planning!

Proposition 1.5.8 (isometry invariance of the Riemannian distance function)**.** *Let* $f \colon (M_1, g_1) \longrightarrow (M_2, g_2)$ *be a local isometry. Then:*

$$\forall_{x,y \in M_1} \quad d_{g_2}\bigl(f(x), f(y)\bigr) \leq d_{g_1}(x, y)$$

In particular: If f *is an isometry, then* f *is isometric with respect to the Riemannian distance functions* d_{g_1} *and* d_{g_2}*, i.e.,*

$$\forall_{x,y \in M_1} \quad d_{g_2}\bigl(f(x), f(y)\bigr) = d_{g_1}(x, y).$$

Proof. The second claim follows directly from the first one (applied to f and its inverse). We now show the first claim: Let $\gamma \colon I \longrightarrow M_1$ be a piecewise regular curve from x to y. Then the isometry invariance of the length

(Proposition 1.5.5) shows that

$$\begin{aligned} d_{g_2}\bigl(f(x), f(y)\bigr) &\leq L_{g_2}(f \circ \gamma) && (\text{Why?}) \\ &= L_{g_1}(\gamma). && (f \text{ is a local isometry; Proposition 1.5.5}) \end{aligned}$$

Taking the infimum over all piecewise regular curves on M_1 from x to y thus shows that $d_{g_2}(f(x), f(y)) \leq d_{g_1}(x, y)$. □

Caveat 1.5.9. Local isometries in general do *not* preserve the Riemann distance function! For example, this can be seen by looking at the exponential map $\mathbb{R} \longrightarrow \mathbb{S}^1$ from Example 1.3.20 (check!).

Proposition 1.5.10 (recovering the Euclidean metric)**.** *Let $n \in \mathbb{N}$. Then the Riemannian distance function associated with the Euclidean Riemannian metric on $\mathbb{R}^n$ equals the Euclidean metric on $\mathbb{R}^n$.*

Proof. Let g denote the Euclidean Riemannian metric on $\mathbb{R}^n$, let d_g be the associated Riemannian distance function, and let d denote the usual Euclidean metric on $\mathbb{R}^n$. Moreover, let $x, y \in \mathbb{R}^n$. In view of Proposition 1.5.8 and as translations and $\mathrm{O}(n)$ act both by Riemannian and Euclidean isometries on $\mathbb{R}^n$ (Example 1.3.20), we may assume without loss of generality that x and y lie both on the first axis.

- We have $d_g(x, y) \leq d(x, y)$: This can be seen by looking at the straight line segments in $\mathbb{R}^n$ from x to y (Example 1.5.4; check!).

- We have $d_g(x, y) \geq d(x, y)$: Let $\gamma\colon [a, b] \longrightarrow \mathbb{R}^n$ be a piecewise regular curve from x to y. We now use the orthogonal projection $\pi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ onto the first coordinate axis to show that $L_g(\gamma) \geq d(x, y)$:

  As π is smooth, the composition $\pi \circ \gamma$ is piecewise smooth. Moreover, $\mathrm{d}\,\pi$ can at every point be identified with π and $\|\pi(v)\|_2 \leq \|v\|_2$ holds for all $v \in \mathbb{R}^n$. Therefore, we obtain

$$\begin{aligned} L_g(\gamma) &= \int_a^b \bigl\|\gamma'(t)\bigr\|_2 \, dt \geq \int_a^b \bigl\|\pi \circ \gamma'(t)\bigr\|_2 \, dt = \int_a^b \bigl\|(\pi \circ \gamma)'(t)\bigr\|_2 \, dt \\ &= \int_a^b |f'(t)| \, dt && (\text{with } f := \pi \circ \gamma\colon [a, b] \to \mathbb{R}) \\ &\geq \left| \int_a^b f'(t) \, dt \right| = \bigl|f(b) - f(a)\bigr| \\ &= \bigl\|\gamma(b) - \gamma(a)\bigr\|_2 = \|y - x\|_2 && (x \text{ and } y \text{ lie on the first axis}) \\ &= d(x, y). \end{aligned}$$

Hence, $d_g(x, y) = d(x, y)$. □

Proposition 1.5.11 (length estimate for other Riemannian metrics on Euclidean space). *Let $n \in \mathbb{N}$, let $U \subset \mathbb{R}^n$ be open, let g be a Riemannian metric on U, and let $K \subset U$ be compact. Then, there exist $c, C \in \mathbb{R}_{>0}$ with*

$$\forall_{x \in K} \quad \forall_{v \in \mathrm{T}_x U} \quad c \cdot \|v\|_2 \leq \|v\|_{g_x} \leq C \cdot \|v\|_2.$$

Proof. This is a standard compactness argument: The set

$$S := \bigl\{ v \in \mathrm{T}_x U \bigm| x \in K,\ \|v\|_2 = 1 \bigr\} \subset \mathrm{T}\, U \cong_{\mathsf{VectB}} U \times \mathbb{R}^n$$

is compact (check!) and the function

$$\begin{aligned} f \colon \mathrm{T}\, U &\longrightarrow \mathbb{R} \\ \mathrm{T}_x U \ni v &\longmapsto \|v\|_{g_x} \end{aligned}$$

is continuous. Moreover, clearly, f is positive on S. Therefore, $f|_S$ attains a minimum $c \in \mathbb{R}_{>0}$ and maximum $C \in \mathbb{R}_{>0}$.

Then, for all $x \in K$ and all $v \in \mathrm{T}_x U \setminus \{0\}$, we obtain

$$\begin{aligned} c \cdot \|v\|_2 &= c \cdot \|v\|_2 \cdot \Bigl\| \frac{1}{\|v\|_2} \cdot v \Bigr\|_2 \leq \|v\|_2 \cdot \Bigl\| \frac{1}{\|v\|_2} \cdot v \Bigr\|_{g_x} \\ &= \|v\|_{g_x} \end{aligned}$$

and, similarly, $\|v\|_{g_x} \leq C \cdot \|v\|_2$. Moreover, for 0, the estimates are satisfied anyway. □

Corollary 1.5.12 (the local Euclidean estimate). *Let (M, g) be a Riemannian manifold, let $x \in M$, and let $U \subset M$ be an open neighbourhood of x. Moreover, let d denote the Euclidean metric on $\mathbb{R}^n$. Then, there exists a smooth chart $\varphi \colon V \longrightarrow V'$ of M around x with $\overline{V} \subset U$ and $C, D \in \mathbb{R}_{>0}$ with the following properties:*

1. *For all $y \in V$, we have*

$$d_g(x, y) \leq C \cdot d\bigl(\varphi(x), \varphi(y)\bigr).$$

2. *For all $y \in M \setminus \overline{V}$, we have*

$$d_g(x, y) \geq D.$$

Proof. We choose V as a suitable pre-image of a Euclidean ball under a smooth chart and then apply our knowledge on Riemannian metrics and distances on Euclidean space:

Choice of V. Let $\varphi \colon W \longrightarrow W'$ be a smooth chart of M around x with $W \subset U$. As $W' \subset \mathbb{R}^n$ is an open neighbourhood of $\varphi(x)$, there exists an $r \in \mathbb{R}_{>0}$ with $\overline{B}_r(x) \subset W'$. We then set

$$V := \varphi^{-1}\bigl(B_r(\varphi(x))\bigr) \subset W \subset U.$$

Moreover, let $\overline{g} := (\varphi^{-1})^* g|_W$ be the pull-back Riemannian metric on W'. As $\overline{B}_r(\varphi(x))$ is compact, there exist $c, C \in \mathbb{R}_{>0}$ with

$$\forall_{z \in \overline{B}_r(\varphi(x))} \quad \forall_{v \in \mathrm{T}_z W'} \quad c \cdot \|v\|_2 \leq \|v\|_{\overline{g}_z} \leq C \cdot \|v\|_2$$

(Proposition 1.5.11). Hence, if γ is a piecewise regular curve in V, we obtain (by monotonicity of the integral)

$$c \cdot L_{\text{Euclidean}}(\varphi \circ \gamma) \leq L_g(\gamma) \leq C \cdot L_{\text{Euclidean}}(\varphi \circ \gamma).$$

Ad 1. Let $y \in \overline{V}$ and let γ be the φ^{-1}-image of the straight line segment from $\varphi(x)$ to $\varphi(y)$. As $B_r(\varphi(x))$ is convex, the curve γ is contained in V. Moreover, γ is regular. Therefore, we obtain

$$\begin{aligned}
d_g(x,y) &\leq L_g(\gamma) && \text{(by defintion of } L_g\text{)}\\
&\leq C \cdot L_{\text{Euclidean}}(\varphi \circ \gamma) && \text{(see above)}\\
&= C \cdot L_{\text{Euclidean}}(\text{straight line segment from } \varphi(x) \text{ to } \varphi(y))\\
&= C \cdot d\bigl(\varphi(x), \varphi(y)\bigr). && \text{(Example 1.5.4)}
\end{aligned}$$

Ad 2. Let $y \in M \setminus V$ and let $\gamma\colon [a,b] \longrightarrow M$ be a piecewise regular curve from x to y. We then consider the first exit time of V:

$$t_0 := \inf\bigl\{t \in [a,b] \bigm| \gamma(t) \notin V\bigr\} \in [a,b]$$

By construction, $\gamma([a,t_0]) \subset \overline{B}_r(\varphi(x))$ and

$$d\bigl(\varphi(x), \varphi(\gamma(t_0))\bigr) = r.$$

Therefore, we obtain

$$\begin{aligned}
L_g(\gamma) &\geq L_g(\gamma|_{[a,t_0]})\\
&\geq c \cdot L_{\text{Euclidean}}(\varphi \circ \gamma|_{[a,t_0]}) && \text{(see above)}\\
&\geq c \cdot d\bigl(\varphi(x), \varphi(\gamma(t_0))\bigr) && (\text{Why?} \qquad)\\
&= c \cdot r.
\end{aligned}$$

Taking the infimum over all piecewise regular curves from x to y shows that $d_g(x,y) \geq c \cdot r$. □

Theorem 1.5.13 (the metric of a Riemannian metric)**.** *Let (M,g) be a connected Riemannian manifold.*

1. *Then the Riemannian distance function associated with g is a metric on M.*

2. *Moreover, the topology on M induced by the Riemannian distance function coincides with the original topology.*

Proof. Let d_g denote the Riemannian distance function associated with g.

Ad 1. We first check that $d_g(x, y)$ is *finite* for all $x, y \in M$: By definition of d_g, it suffices to show that there exists a piecewise regular curve from x to y on M. As M is connected (by assumption) and locally path-connected (as a manifold), M is also path-connected. Let $\gamma\colon [a, b] \longrightarrow M$ be a continuous path from x to y. Because $[a, b]$ is compact, there exists a partition $a = a_0 < \dots < a_k = b$ of $[a, b]$ with the property that each of the $\gamma([a_j, a_{j+1}])$ lies in the preimage of a Euclidean ball under a smooth chart φ_j (check!). Then, applying φ^{-1} to the straight line segments from $\varphi(\gamma(a_j))$ to $\varphi(\gamma(a_{j+1})$ gives a regular curve γ_j in M from $\gamma(a_j)$ to $\gamma(a_{j+1})$. Thus, the concatenation of $\gamma_0, \dots, \gamma_{k-1}$ is a piecewise regular curve $\overline{\gamma}$ from x to y. In particular,

$$d_g(x, y) \le L_g(\overline{\gamma}) < \infty.$$

It now remains to establish symmetry, positive definiteness, and the triangle inequality.

Symmetry and the *triangle inequality* for d_g follow by reverting and concatenating piecewise regular curves on M (check!).

For *positive definiteness*, we may argue as follows: Let $x, y \in M$ with $x \neq y$. We then choose an open neighbourhood $U \subset M$ of x with $y \notin \overline{U}$. Let $V \subset U$ be a neighbourhood as provided by Corollary 1.5.12, with associated constant $D \in \mathbb{R}_{>0}$. In particular,

$$d_g(x, y) \ge D > 0.$$

Ad 2. We separate the proof into the two directions:

- Let $U \subset M$ be open in the manifold topology. We show that U is also open with respect to the metric topology of d_g: So, let $x \in U$. Let $V \subset U$ be a neighbourhood as provided by Corollary 1.5.12, with associated constant $D \in \mathbb{R}_{>0}$. Then,

  $$B_D^{M,d_g}(x) \subset V.$$

  Thus, U is open in the metric topology of d_g.

- Conversely, let $U \subset M$ be open in the metric topology of d_g. We show that U is open in the manifold topology: Let $x \in U$. As U is d_g-open, there exists an $r \in \mathbb{R}_{>0}$ with

  $$B_r^{M,d_g}(x) \subset U.$$

  Let $\varphi\colon V \longrightarrow V'$ be a smooth chart of M around x, as provided by Corollary 1.5.12 (applied to the open neighbourhood M of x in M), with associated constant $C \in \mathbb{R}_{>0}$. Then

$$V \cap \varphi^{-1}\big(B_{1/2 \cdot r/C}(\varphi(x))\big) \subset B_r^{M,d_g}(x) \subset U$$

(check!). Moreover, this intersection is open in the manifold topology (both intersectants are open in the manifold topology) and contains x. This shows that U is open in the manifold topology.

Hence, the original topology on M coincides with the metric topology induced by the Riemannian distance function d_g. $\square$

By now, we equipped Riemannian manifolds with a metric geometry. However, so far, we are not able to compute the Riemannian distance function beyond simple examples.

In metric geometry, *geodesics* (i.e., isometric embeddings of intervals into the given metric space) play a distinguished role. Geodesics are used to model common geometric shapes (e.g., geodesic triangles) and also serve as a means to describe other geometric phenomena. Similarly, also in Riemannian geometry, geodesics are a key tool and their local versions admit an analytic characterisation in terms of "second derivatives" as straightest curves (which is the same as curves with zero curvature).

However, it turns out that it is rather hard to work directly with the metric notion of geodesics in Riemannian geometry. To make the full power of analytic tools available, it is more convenient to develop these notions in a different order:

- First, introduce a notion of "second derivatives".
- Then, introduce geodesics via a suitable differential equation.

The notion of "second derivatives" requires some thought. In fact, the problems occuring when trying to give a good definition of "second derivative" are intimately related to the notion of curvature. In analytic terms, this is formulated in the language of connections.

Once we have the analytic approach to geodesics at our disposal, we will also be able to compute the Riemannian distance function in more cases, by combining geometry and analysis.

1.5.3 Volume and orientation

Just like an inner product on a vector space induces a notion of volume, also Riemannian metrics lead to a notion of volume of manifolds. We briefly recall basic facts on volume.

We first recall the integration of functions on Riemannian manifolds: Functions are integrated by

- integrating the local contributions, measured with respect to the Riemannian metric, via smooth charts, and then

- summing these contributions via a partition of unity.

This integral is independent of the chosen charts because the rescaling via the Riemannian metric transforms in the correct way.

In a second step, we recall how the volume can be expressed in terms of orientations and integration of forms (which is sometimes the more convenient setup as it immediately allows to use tools such as Stokes's theorem).

Moreover, we briefly indicate how volume growth of balls in Riemannian manifolds can be used to distinguish Riemannian manifolds in some examples.

Remark 1.5.14 (integration of functions on Riemannian manifolds)**.** Let (M, g) be a Riemannian manifold, let $(\varphi_i \colon U_i \to U_i')_{i \in I}$ be a countable smooth atlas for M, and let $(\psi_i)_{i \in I}$ be a partition of unity that is subordinate to $(U_i)_{i \in I}$ (Appendix A.2). Moreover, for each $i \in I$, we let $(g^i_{j,k})_{j,k \in \{1,\dots,n\}}$ denote the Riemannian metric on U_i in the local coordinates given by φ_i (Remark 1.3.8).

- A map $f \colon M \longrightarrow \mathbb{R}$ is *measurable*, if it is Borel measurable.

  This is equivalent to asking that for each $i \in I$, the composition $f \circ \varphi_i^{-1} \colon U_i' \longrightarrow \mathbb{R}$ is measurable (check!).
- A measurable map $f \colon M \longrightarrow \mathbb{R}$ is *integrable* if the sum
$$\sum_{i \in I} \int_{U_i'} \psi_i(x) \cdot |f \circ \varphi_i^{-1}(x)| \cdot \sqrt{\det(g^i(\varphi_i^{-1}(x)))} \; d\lambda^n(x)$$
  of Lebesgue integrals is finite.
- The *integral* of an integrable function $f \colon M \longrightarrow \mathbb{R}$ is defined as
$$\int_M f \; d\mu_{M,g} := \sum_{i \in I} \int_{U_i'} \psi_i \cdot f \circ \varphi_i^{-1} \cdot \sqrt{\det(g^i \circ \varphi_i^{-1})} \; d\lambda^n.$$

All these notions do *not* depend on the chosen smooth atlas or the chosen partition of unity [1, analogous to Chapter 1.6].

This integral has the usual linearity, monotonicity, approximation, and transformation properties. On manifolds, we thus also obtain a corresponding (possibly infinite) measure $\mu_{M,g}$ on the Borel σ-algebra.

Definition 1.5.15 (volume)**.** Let (M, g) be a Riemannian manifold. If the constant function 1 on M is integrable, then the *volume* of (M, g) is defined as
$$\operatorname{vol}(M, g) := \int_M 1 \; d\mu_{M,g} \in \mathbb{R}_{\geq 0}.$$
If the constant function 1 on M is *not* integrable, then we set $\operatorname{vol}(M, g) := \infty$.

Example 1.5.16 (volumes of spheres and balls)**.** Let $n \in \mathbb{N}_{>0}$ and $R \in \mathbb{R}_{>0}$. Then a well-known inductive computation and Fubini's theorem show that (check!):

$$\mathrm{vol}\bigl(\mathbb{S}^n(R)\bigr) = \frac{2 \cdot \pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2}+1)} \cdot R^n$$
$$\mu_{\mathbb{R}^n}\bigl(B_R^{\mathbb{R}^n}(0)\bigr) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \cdot R^n.$$

Here, $\mathbb{S}^n(R)$ carries the round metric of radius R and $\mathbb{R}^n$ carries the Euclidean Riemannian metric.

In addition to direct applications in real-world models, geometric invariants such as the Riemannian distance or volume also should allow us to distinguish Riemannian manifolds. For example, we could ask the following questions about the model spaces: Let $n \in \mathbb{N}$.

- For which radii $R_1, R_2 \in \mathbb{R}_{>0}$ is $\mathbb{S}^n(R_1)$ isometric to $\mathbb{S}^n(R_2)$?
- For which radii $R_1, R_2 \in \mathbb{R}_{>0}$ is $\mathbb{H}^n(R_1)$ isometric to $\mathbb{H}^n(R_2)$?
- Is there a radius $R \in \mathbb{R}_{>0}$ for which $\mathbb{R}^n$ and $\mathbb{H}^n(R)$ are isometric?
- Is there a radius $R \in \mathbb{R}_{>0}$ for which $\mathbb{R}^n$ and $\mathbb{H}^n(R)$ are locally isometric?
- Is there a radius $R \in \mathbb{R}_{>0}$ for which $\mathbb{R}^n$ and $\mathbb{S}^n(R)$ are locally isometric?

Most of these questions require a better understanding of the Riemannian distance function. However, in some cases, we can get away with the volume and rough distance estimates.

Proposition 1.5.17 (isometry invariance of volume). *Isometric Riemannian manifolds have the same volume.*

Proof. Let $f\colon (M,g) \longrightarrow (M',g')$ be an isometry of Riemannian manifolds. Then $f^*g' = g$ and so choosing charts and partitions of unity on M and M', respectively, that are compatible via f, we can use the transformation formula for integration to prove the claim (check!). ☐

Example 1.5.18. Let $n \in \mathbb{N}_{>0}$ and let $R_1, R_2 \in \mathbb{R}_{>0}$ with $R_1 \neq R_2$. Then the round spheres $\mathbb{S}^n(R_1)$ and $\mathbb{S}^n(R_2)$ are *not* isometric because

Definition 1.5.19 (volume growth). Let (M,g) be a Riemannian manifold and let $x \in M$. Then the *volume growth function of* (M,g) *at* x is the function

$$\begin{aligned} \varrho_x^{(M,g)}\colon \mathbb{R}_{>0} &\longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\} \\ r &\longmapsto \mu_{M,g}\bigl(B_r^{(M,d_g)}(x)\bigr) \end{aligned}$$

Proposition 1.5.20 (isometry invariance of volume growth). *Let* (M,g) *and* (M',g') *be Riemannian manifolds, let* $x \in M$*, and let* $f\colon (M,g) \longrightarrow (M',g')$ *be a map.*

1. *If f is an isometry, then* $\varrho_x^{(M,g)} = \varrho_{f(x)}^{(M',g')}$.

2. *If f is a local isometry, then there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that*

$$\forall_{r \in [0,\varepsilon)} \quad \varrho_x^{(M,g)}(r) = \varrho_{f(x)}^{(M',g')}(r).$$

Proof. Ad 1. Isometries map metric balls to metric balls of the same radius because the Riemannian distance is isometry invariant (Proposition 1.5.8). Moreover, the Riemannian volume is isometry invariant (Proposition 1.5.17). Thus, if $f\colon (M,g) \longrightarrow (M',g')$ is an isometry, then we have $f(B_r^{(M,d_g)}(x)) = B_r^{(M',d_{g'})}(f(x))$ and for all $r \in \mathbb{R}_{>0}$:

$$\varrho_x^{(M,g)}(r) = \mu_{M,g}\bigl(B_r^{(M,d_g)}(x)\bigr) = \mu_{M',g'}\bigl(B_r^{(M',d_{g'})}(f(x))\bigr) = \varrho_{f(x)}^{(M',g')}(r).$$

Ad 2. For local isometries, we just restrict to small enough open subsets (which will contain all small enough balls by Proposition 1.5.13) and then apply the first part. □

Example 1.5.21 (volume growth of Euclidean space)**.** Let $n \in \mathbb{N}_{>0}$. Then the volume growth function of $\mathbb{R}^n$ with respect to the Euclidean Riemannian metric is polynomial of degree n (Example 1.5.16).

Example 1.5.22 (volume growth of hyperbolic space)**.** Let $n \in \mathbb{N}_{\geq 2}$. Then a rough distance and volume estimate shows that the volume growth function of $\mathbb{H}^n(R)$ (with respect to the hyperbolic Riemannian metric of radius R) is at least exponential for each $R \in \mathbb{R}_{>0}$ (Exercise). Therefore, the polynomial volume growth of $\mathbb{R}^n$ (Example 1.5.21) and (local) isometry invariance of volume growth (Proposition 1.5.20) show:

- $\mathbb{R}^n$ with the Euclidean Riemannian metric is *not* isometric to $\mathbb{H}^n(R)$.
- $\mathbb{R}^n$ with the Euclidean Riemannian metric is *not* locally isometric to $\mathbb{H}^n(R)$.

However, beyond simple examples, we cannot handle volume growth well because we cannot yet compute enough examples of the Riemannian distance function (which we need to describe the metric balls).

Volume growth builds an interesting bridge between Riemannian geometry and (geometric) group theory. We will return to this aspect later (Chapter 4.6).

Alternatively, one can express volume and integration also in terms of differential forms, using orientations: An orientation of a manifold is just a smoothly parametrised family of orientations of the tangent spaces.

Remark 1.5.23 (orientation of a vector space)**.** Let V be an $\mathbb{R}$-vector space of finite dimension $n \in \mathbb{N}$. Two bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ of V *have* *the same orientation* if the base change map, i.e., the $\mathbb{R}$-linear map uniquely determined by

$$\begin{aligned} V &\longrightarrow V \\ v_j &\longmapsto w_j, \end{aligned}$$

has *positive* determinant.

As the determinant provides an $\mathbb{R}$-linear isomorphism $\bigwedge^n V \longrightarrow \mathbb{R}$, we know that bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ have the same orientation if and only if there exists a $\lambda \in \mathbb{R}_{>0}$ with

$$v_1 \wedge \dots \wedge v_n = \lambda \cdot (w_1 \wedge \dots \wedge w_n)$$

in $\bigwedge^n V$.

If $n > 0$, then V admits exactly two different classes of orientations through bases (check!).

Definition 1.5.24 (orientability, orientation of a smooth manifold)**.** Let M be a smooth manifold of dimension n.

- An *orientation of M* is a section $\omega \in \Gamma(\bigwedge^n(\mathrm{T}\, M))$ with the property that
  $$\forall_{x \in M} \quad \omega(x) \neq 0.$$
  We call M *orientable* if there exists an orientation of M.
- Two orientations on M are *equivalent* if they induce the same orientation on all tangent spaces.
- An *oriented* manifold is a pair consisting of a smooth manifold and an (equivalence class of an) orientation on this manifold.

Definition 1.5.25 (orientation-preserving)**.** A smooth map $M \longrightarrow N$ between oriented smooth manifolds is *orientation-preserving* if it pulls back the orientation on N to an orientation equivalent to the orientation on M. If it pulls back the orientation on N to an orientation that is not equivalent to the orientation on M, it is called *orientation-reversing*.

Example 1.5.26. Let $n \in \mathbb{N}$. We consider $\mathbb{R}^n$ with the orientation given by the standard basis. Then an $\mathbb{R}$-linear isomorphism $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is orientation-preserving if and only if Hint

Proposition 1.5.27 (volume form [1, Chapter 3.7.2])**.** *Let (M, g) be an oriented Riemannian manifold of dimension n with orientation $\omega \in \Gamma(\bigwedge^n(\mathrm{T}\, M))$. Then there is a unique $\mathrm{Vol}_{M,g} \in \Omega^n(M)$, the* volume form*, with the following property: For each orientation-preserving smooth chart $U \longrightarrow U'$ with associated local coordinates $x^1, \dots, x^n \colon U \longrightarrow \mathbb{R}$, we have*

$$\forall_{x \in U} \quad \mathrm{Vol}_{M,g}(x) = \sqrt{\det(g_{ij}(x))_{i,j}} \cdot \mathrm{d}\, x^1 \wedge \dots \wedge \mathrm{d}\, x^n,$$

where $(g_{ij})_{i,j\in\{1,\dots,n\}}$ *are the coefficients of the Riemannian metric* g *in the local coordinates* $x^1, \dots, x^n$ *(Remark 1.3.8).*

Proposition 1.5.28 (volume via volume form)**.** *Let* (M, g) *be an oriented Riemannian manifold. Then*

$$\mathrm{vol}(M, g) = \int_M \mathrm{Vol}_{M,g}\,.$$

Proof. By definition, $\mathrm{vol}(M, g) = \int_M 1\, d\mu_{M,g}$. We now only need to compare the definition of this integral with the definition of the integral over the form $\mathrm{Vol}_{M,g}$ via partitions of unity (check!). □

Study note. Write a summary of Chapter 1, keeping the following questions in mind:

- What are smooth manifolds?
- What are smooth vector bundles? What are important examples of smooth vector bundles?
- What are Riemannian manifolds? What is the geometric motivation behind this definition?
- What are typical examples of Riemannian manifolds?
- How can one work with concrete examples?
- How can one work in local coordinates?
- What notions of symmetry can be considered?
- How can one define the Riemannian distance function and the Riemannian volume?

2

Curvature: Foundations

Curvature of geometric objects quantifies the amount of "bending" in comparison with "flat" Euclidean space. There are several approaches to formalising curvature. We will follow the classical analytic approach that is based on the curvature of curves. The main analytic challenge in this strategy is to find an intrinsic notion of "acceleration" of curves.

The "acceleration" of curves can be handled through a suitable notion of directional derivatives of sections of smooth vector bundles, via so-called connections. In the presence of a Riemannian metric, there is a canonical connection on the tangent bundle.

In this chapter, we will develop the notion of connections and the associated notions of geodesics and parallel transport. Moreover, we will use connections to introduce the Riemannian curvature tensor as well as sectional, Ricci, and scalar curvature.

In the follow-up chapters, we will then have a closer look at Riemannian geodesics and the interaction between curvature properties and global shape.

Overview of this chapter.

Running example. Euclidean space, submanifolds of Euclidean space

2.1 The idea of curvature

Curvature of geometric objects quantifies the amount of “bending” in comparison with “flat” Euclidean space. Intuitively, spherical shapes should correspond to positive curvature, flat shapes to vanishing curvature, and saddle shapes to negative curvature (Figure 2.1).

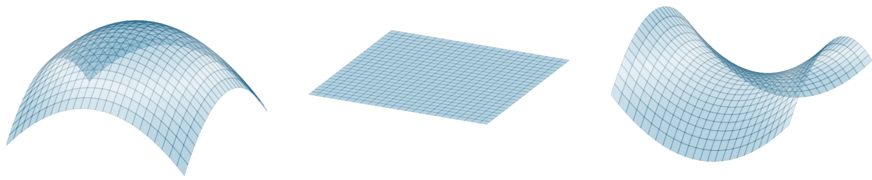


Figure 2.1.: Positive, vanishing, negative curvature, schematically

There are several approaches to curvature:

① Via curvature of curves.

② Via angles in geodesic triangles.

③ Via local and global growth rates of volumes of balls.

These three approaches are related to each other and all of them need an efficient handling of geodesics.

In this course, we will follow approach ① and only briefly indicate the interaction with the approaches ② and ③.

In order to use the curvature of curves as basic building block of curvature of high-dimensional geometric objects one proceeds in the following steps:

- The curvature of plane or spatial curves is measured in terms of the “acceleration”, i.e., in terms of the second derivative.
- The curvature of a Riemannian surface at a given point is measured as the product of the minimal and maximal (signed) curvature of all curves on this surface passing through this point.
- The curvature of a Riemannian manifold at a given point is measured as the curvatures of all tangential surfaces passing through this point. In particular, at each point, there will be many curvatures and there are several ways to combine/organise these curvatures.

The main analytic challenge in this strategy is to find an intrinsic notion of "acceleration" of curves and then to capture the second and third steps in terms of appropriate tensors.

Two naive approaches to finding a definition of second derivatives of curves are:

① A local approach via second derivatives in local coordinates/charts: Let $\gamma\colon I \longrightarrow M$ be a smooth curve, let $t \in I$, let $\varphi\colon U \longrightarrow U'$ be a smooth chart for M around $\gamma(t)$, let $x^1, \dots, x^n\colon M \longrightarrow \mathbb{R}$ denote the coordinates induced by φ, and let $\gamma^j_\varphi := x^j \circ \gamma|_U$ for all $j \in \{1, \dots, n\}$. Then, we can write $\gamma|_U = \varphi^{-1} \circ \varphi \circ \gamma|_U = \varphi^{-1} \circ (\gamma^1_\varphi, \dots, \gamma^n_\varphi)$. Therefore (with $\partial^\varphi_j := \mathrm{d}\,\varphi^{-1}(e_j)$),

$$\mathrm{d}_t\,\gamma = \mathrm{d}_{\gamma(t)}\,\varphi^{-1}\bigl({\gamma^1_\varphi}'(t), \dots, {\gamma^n_\varphi}'(t)\bigr) = \sum_{j=1}^n {\gamma^j_\varphi}'(t) \cdot \partial^\varphi_j .$$

Analogously, we could consider the expression

$$\sum_{j=1}^n {\gamma^j_\varphi}''(t) \cdot \partial^\varphi_j .$$

② A global approach via iterated tangent bundles: If $\gamma\colon I \longrightarrow M$ is a smooth curve, defined on an open interval I, then $\mathrm{d}\,\gamma$ is a smooth map $\mathrm{T}\,I \longrightarrow \mathrm{T}\,M$, and $\mathrm{d}(\mathrm{d}\,\gamma)$ is a smooth map $\mathrm{T}(\mathrm{T}\,I) \longrightarrow \mathrm{T}(\mathrm{T}\,M)$.

The problem with the second idea is that the result should be something of type $I \longrightarrow \mathrm{T}\,M$ (and not something involving iterated tangent bundles).

The problem with the first idea is that it does *not* directly lead to tangent vectors that are independent of the chosen chart because of the wrong transformation behaviour with respect to the change of charts:

Example 2.1.1 (non-intrinsic acceleration)**.** On the smooth manifold $\mathbb{R}^2$, we consider the smooth arc

$$\begin{aligned} \gamma\colon (0,\pi) &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (\cos t, \sin t). \end{aligned}$$

Let $\varphi = \mathrm{id}_{\mathbb{R}^2}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the straight coordinates and let

$$\begin{aligned} \psi\colon \mathbb{R}^2 \setminus \{0\} &\longrightarrow \mathbb{R}_{>0} \times [0, 2\cdot\pi) \\ (r \cdot \cos\vartheta, r \cdot \sin\vartheta) &\longleftarrow (r, \vartheta) \end{aligned}$$

be the polar coordinate transformation. Then we obtain:

- In the straight local coordinates determined by φ, we have $\gamma^1_\varphi = \cos$ and $\gamma^2_\varphi = \sin$. Therefore,

$$\gamma_\varphi^1{}'' = -\cos \quad \text{and} \quad \gamma_\varphi^2{}'' = -\sin .$$

- In the polar coordinates ψ, we have $\gamma_\psi^1 = 1$ and $\gamma_\psi^2 = (t \mapsto t)$. Therefore,

$$\gamma_\psi^1{}'' = 0 = \gamma_\psi^2{}''.$$

In particular, for these charts, for all $t \in (0, \pi)$:

$$\sum_{j=1}^{2} \gamma_\varphi^j{}''(t) \cdot \partial_j^\varphi \neq 0 = \sum_{j=1}^{2} \gamma_\psi^j{}''(t) \cdot \partial_j^\psi .$$

This should not come as a surprise: With ψ, we packed the "bending" into the chart/coordinates and the representation in the tangent space is not able to compensate for this second derivative.

In both cases, the problem can be resolved by using a Riemannian metric to add correction terms and extract the geometrically relevant part of the second derivative. It is possible to formulate this extraction in both these approaches. However, it will be more convenient in the long run to resort to a more general and more axiomatic approach:

③ We will formulate a wishlist for the differentiation of sections of smooth vector bundles and show that there is a canonical way to satisfy these axioms in the presence of a Riemannian metric.

This leads to the notion of *connections* on smooth vector bundles. We will first introduce the general setup for connections and only in a second step we will add the desired interaction with the Riemannian metric. Finally, this will allow us to introduce the Riemannian curvature tensor and its derivates: sectional, Ricci, and scalar curvature.

2.2 Connections

Connections formalise the notion of differentiation of sections of smooth vector bundles in the direction of tangent vector fields on the base manifold.

Such a differentiation operator should be

- C^∞-linear in the direction argument,
- it should be linear in the section argument, and
- it should satisfy a product rule for the multiplication with smooth functions in the section argument.

In particular, connections will allow us to differentiate the velocity vector field along a smooth curve in a reasonable way and thus to introduce geodesics and parallel transport (with respect to the given connection). Parallel transport will finally also explain in which sense connections "connect" nearby tangent spaces.

All of this is still an abstract setup that does not require a Riemannian metric. In Chapter 2.3, we will add the Riemannian aspect and thus turn this into geometrically meaningful constructions.

2.2.1 Connections

We now introduce the notion of connections, we will give the standard Euclidean example, and then we will show (analogously to the case of Riemannian metrics) that every smooth manifold admits a connection on its tangent bundle.

Definition 2.2.1 (connection). Let $\pi\colon E \longrightarrow M$ be a smooth vector bundle over a smooth manifold M. A *connection on E (or π)* is a map "nabla" ∇ of type

$$\nabla\colon \Gamma(\mathrm{T}\, M) \times \Gamma(E) \longrightarrow \Gamma(E)$$

with the following properties:

(FL1) *C^∞-linearity in the first argument:* For all $X_1, X_2 \in \Gamma(\mathrm{T}\, M)$, all $Y \in \Gamma(E)$, and all $f_1, f_2 \in C^\infty(M)$, we have

$$\nabla_{f_1 \cdot X_1 + f_2 \cdot X_2} Y = f_1 \cdot \nabla_{X_1} Y + f_2 \cdot \nabla_{X_2} Y.$$

(L2) *$\mathbb{R}$-linearity in the second argument:* For all $X \in \Gamma(\mathrm{T}\, M)$, all $Y_1, Y_2 \in \Gamma(E)$, and all $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\nabla_X (\lambda_1 \cdot Y_1 + \lambda_2 \cdot Y_2) = \lambda_1 \cdot \nabla_X Y_1 + \lambda_2 \cdot \nabla_X Y_2.$$

(F2) *Product rule in the second argument:* For all $X \in \Gamma(\mathrm{T}\, M)$, all $Y \in \Gamma(E)$, and all $f \in C^\infty(M)$, we have

$$\nabla_X (f \cdot Y) = X(f) \cdot Y + f \cdot \nabla_X Y.$$

Here, $X(f)$ has the following meaning: For all $x \in M$, the tangent vector $X(x)$ can be viewed as a derivation (Remark 1.1.25) and thus can be applied to f via $(X(f))(x) := \partial_{X(x)}(f)$.

Connections on the tangent bundle $\mathrm{T}\, M \longrightarrow M$ are also called *linear connections* or *affine connections*.

Definition 2.2.2 (covariant derivative). Let ∇ be a connection on a smooth vector bundle $E \longrightarrow M$ over a smooth manifold M. If $X \in \Gamma(\mathrm{T}\, M)$ and

$Y \in \Gamma(E)$, then $\nabla_X Y$ is called the *covariant derivative of Y in the direction of X (with respect to ∇).*

Example 2.2.3 (the Euclidean connection)**.** Let $n \in \mathbb{N}$. Using the standard description of $\mathrm{T}\,\mathbb{R}^n$ via the partial derivatives $\partial_1, \dots, \partial_n$, we can define

$$\begin{aligned} \overline{\nabla} \colon \Gamma(\mathrm{T}\,\mathbb{R}^n) \times \Gamma(\mathrm{T}\,\mathbb{R}^n) &\longrightarrow \Gamma(\mathrm{T}\,\mathbb{R}^n) \\ (X, Y) &\longmapsto \sum_{j=1}^{n} X(Y^j) \cdot \partial_j, \end{aligned}$$

where $Y^1, \dots, Y^n$ are the coordinates of Y with respect to $(\partial_1, \dots, \partial_n)$. A straightforward computation shows that $\overline{\nabla}$ is indeed a linear connection on $\mathbb{R}^n$ (check!), the *Euclidean connection on* $\mathbb{R}^n$.

Proposition 2.2.4 (pulling back linear connections)**.** *Let $\varphi \colon M \longrightarrow N$ be a diffeomorphism between smooth manifolds and let ∇ be a linear connection on N. Then*

$$\begin{aligned} \varphi^*\nabla \colon \Gamma(\mathrm{T}\,M) \times \Gamma(\mathrm{T}\,M) &\longrightarrow \Gamma(\mathrm{T}\,M) \\ (X, Y) &\longmapsto \mathrm{d}\,\varphi^{-1} \circ \nabla_{\varphi_* X}(\varphi_* Y) \end{aligned}$$

is a linear connection on M. Here, the pushforward $\varphi_* X \in \Gamma(\mathrm{T}\,N)$ of $X \in \Gamma(\mathrm{T}\,M)$ is defined as *the only reasonable choice* .

Proof. This is a straightforward computation (Exercise). □

Proposition 2.2.5 (convex combinations of connections)**.** *Let M be a smooth manifold, let $E \longrightarrow M$ be a smooth vector bundle over M, let ∇^1, ∇^2 be connections on E, and let $f_1, f_2 \in C^\infty(M)$ with $f_1 + f_2 = 1$. Then*

$$\begin{aligned} f_1 \cdot \nabla^1 + f_2 \cdot \nabla^2 \colon \Gamma(\mathrm{T}\,M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (X, Y) &\longmapsto f_1 \cdot \nabla^1_X Y + f_2 \cdot \nabla^2_X Y \end{aligned}$$

is a connection on E.

Proof. The conditions (FL1) and (L2) are easily verified (check!). We now establish (F2): Let $X \in \Gamma(\mathrm{T}\,M)$, $Y \in \Gamma(E)$, and let $f \in C^\infty(M)$. Then, we have

$$\begin{aligned} (f_1 \cdot \nabla^1 + f_2 \cdot \nabla^2)_X (f \cdot Y) &= f_1 \cdot \nabla^1_X (f \cdot Y) + f_2 \cdot \nabla^2_X (f \cdot Y) \\ &= f_1 \cdot \big(X(f) \cdot Y + f \cdot \nabla^1_X Y\big) + f_2 \cdot \big(X(f) \cdot Y + f \cdot \nabla^2_X Y\big) \\ &= (f_1 + f_2) \cdot X(f) \cdot Y + f \cdot (f_1 \cdot \nabla^1_X Y + f_2 \cdot \nabla^2_X Y) \\ &= 1 \cdot X(f) \cdot Y + f \cdot (f_1 \cdot \nabla^1 + f_2 \cdot \nabla^2)_X Y, \end{aligned}$$

which proves the product rule (F2). □

Quick check 2.2.6. Are (pointwise) $\mathbb{R}$-linear combinations of connections on the same smooth vector bundle again connections?
Yes No

Theorem 2.2.7 (existence of linear connections). *Let M be a smooth manifold. Then M admits a linear connection.*

Study note. Before reading the proof below, you should recall the proof of the existence of Riemannian metrics (Theorem 1.3.6) and try to come up with a proof of Theorem 2.2.7 yourself!

Proof. We proceed as in the proof of the existence of Riemannian metrics (Theorem 1.3.6): Locally, we use the Euclidean connection; we then glue these local connections via a partition of unity (Appendix A.2):

Let $(\varphi_i \colon U_i \to U_i')_{i \in I}$ be a family of charts that covers M.

- *Local situation.* For each $i \in I$, the chart $\varphi_i \colon U_i \longrightarrow U_i'$ is a diffeomorphism. If $\overline{\nabla}^i$ denotes the Euclidean connection on U_i' (Example 2.2.3), then the pullback $\varphi_i^* \overline{\nabla}^i$ is a linear connection on U_i (Proposition 2.2.4).
- *Global glueing.* Let $(\psi_i)_{i \in I}$ be a partition of unity of M subordinate to $(U_i)_{i \in I}$ (Theorem A.2.1). Then

$$\sum_{i \in I} \psi_i \cdot \varphi_i^* \overline{\nabla}^i$$

  is a linear connection on M (Proposition 2.2.5); strictly speaking, Proposition 2.2.5 is not directly applicable to this situation, but the same arguments work (check!).

In particular, M admits a linear connection. $\square$

Example 2.2.8 (induced connection on submanifolds). Let $N \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a closed smooth submanifold. Then the Euclidean connection $\overline{\nabla}$ on $\mathrm{T}\,\mathbb{R}^N$ induces a connection $\nabla^\top$ on $\mathrm{T}\,M$ as follows: For $X, Y \in \Gamma(\mathrm{T}\,M)$, we set

$$\nabla^\top_X Y := p(\overline{\nabla}_{\widetilde{X}} \widetilde{Y}),$$

where $p \colon (\mathrm{T}\,\mathbb{R}^n)|_M \longrightarrow \mathrm{T}\,M$ is the map induced by the orthogonal projection onto the geometric tangent spaces of $M \subset \mathbb{R}^N$ (with respect to the Euclidean inner product on $\mathbb{R}^N$) and where $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathrm{T}\,\mathbb{R}^N)$ are extensions of X and Y, respectively. If M is not closed, we only extend to an open neighbourhood of M and then use locality of connections (Proposition 2.2.11).

Then, $\nabla^\top$ is indeed well-defined (Exercise) and a linear connection on M (Exercise).

This connection has the following geometric interpretation: If $\gamma\colon I \longrightarrow M$ is a smooth curve in M, then $\nabla^{\top}_{\dot\gamma(t)}(\dot\gamma(t))$ selects the contribution of the second derivative $\gamma''(t) \in \mathbb{R}^N$ that is tangential to M (check!).

2.2.2 Local descriptions of connections

As with all items in Riemannian geometry, it is convenient to have a description of connections in local coordinates. For this to make sense, we first have to establish that connections indeed are local operators.

Proposition 2.2.9 (locality of connections). *Let M be a smooth manifold, let $E \longrightarrow M$ be a smooth vector bundle, let ∇ be a connection on E, and let $x \in M$. Then $(\nabla_X Y)(x)$ depends only on the value $X(x)$ and on the restriction of Y to an arbitrarily small neighbourhood of x, i.e.:*

1. *For all $X \in \Gamma(\mathrm{T}\,M)$, all open neighbourhoods $U \subset M$ of x and all $Y_1, Y_2 \in \Gamma(E)$ with $Y_1|_U = Y_2|_U$, we have*
$$(\nabla_X Y_1)(x) = (\nabla_X Y_2)(x).$$
2. *For all $X_1, X_2 \in \Gamma(\mathrm{T}\,M)$ with $X_1(x) = X_2(x)$ and all $Y \in \Gamma(E)$, we have*
$$(\nabla_{X_1} Y)(x) = (\nabla_{X_2} Y)(x).$$

Proof. Ad 1. Let $X \in \Gamma(\mathrm{T}\,M)$. In view of the linearity property (L2), it suffices to show $(\nabla_X Y)(x) = 0$ for all $Y \in \Gamma(E)$ with $Y|_U = 0$ (check!).

Let $Y \in \Gamma(E)$ with $Y|_U = 0$. In order to get something that vanishes on all of M, we choose a bump function; let $f \in C^\infty(M)$ with $\operatorname{supp} f \subset U$ and $f(x) = 1$ (such a bump function does exist). Then, we obtain $f \cdot Y = 0$ and so

$$\begin{aligned}
0 &= (\nabla_X 0)(x) && \text{(by (L2))}\\
&= (\nabla_X (f \cdot Y))(x) && \text{(because } f\cdot Y = 0)\\
&= X(f)(x) \cdot Y(x) + f(x) \cdot (\nabla_X Y)(x) && \text{(by (F2))}\\
&= 0 + 1 \cdot (\nabla_X Y)(x), && \text{(because } Y(x) = 0 \text{ and } f(x) = 1)
\end{aligned}$$

as claimed.

Ad 2. Let $Y \in \Gamma(E)$. In view of the linearity property (FL1), it suffices to show $(\nabla_X Y)(x) = 0$ for all $X \in \Gamma(\mathrm{T}\,M)$ with $X(x) = 0$ (check!).

Let $X \in \Gamma(\mathrm{T}\,M)$ with $X(x) = 0$. Moreover, let $U \subset M$ be a smooth chart neighbourhood of x with coordinates $x^1, \dots, x^n \colon U \longrightarrow \mathbb{R}$ and let $f \in C^\infty(M)$ be a bump function with $\operatorname{supp} f \subset U$ and $f(x) = 1$. Then, we can write X locally in the form

$$X|_U = \sum_{j=1}^{n} X^j \cdot E_j,$$

where $(E_1, \dots, E_n)$ is the dual basis of $(\mathrm{d}\, x^1, \dots, \mathrm{d}\, x^n)$ (using the canonical isomorphism $\mathbf{T}^{0,1}\, M \cong \mathrm{T}\, M$; Remark 1.2.27). Then, we can view $f \cdot E_1, \dots, f \cdot E_n$ as sections of $\mathrm{T}\, M$ and obtain

$$X = \sum_{j=1}^{n} X^j \cdot f \cdot E_j + (1 - f) \cdot X$$

and thus

$$\begin{aligned}
(\nabla_X T)(x) &= \sum_{j=1}^{n} X^j(x) \cdot (\nabla_{f \cdot E_j} Y)(x) + (1 - f)(x) \cdot (\nabla_X Y)(x) \quad \text{(by (FL1))} \\
&= \sum_{j=1}^{n} 0 \cdot (\nabla_{f \cdot E_j} Y)(x) + 0 \cdot (\nabla_X Y)(x) \\
&= 0.
\end{aligned}$$

This completes the proof. □

Quick check 2.2.10. In the proof of the first part of Proposition 2.2.9, in the very last step, we only used $Y(x) = 0$. Why isn't this condition enough for the whole proof?

Hint

Proposition 2.2.11 (restrictions of connections)**.** *Let M be a smooth manifold, let $\pi\colon E \longrightarrow M$ be a smooth vector bundle over$M$, let $U \subset M$ be an open subset, and let ∇ be a connection on E. Then, there exists a unique connection ∇^U on $\pi|_{\pi^{-1}(U)} \colon \pi^{-1}(U) \longrightarrow U$ with*

$$\forall_{X \in \Gamma(\mathrm{T}\, M)} \quad \forall_{Y \in \Gamma(E)} \quad \nabla^U_{X|_U}(Y|_U) = (\nabla_X Y)|_U.$$

Proof. Once one has figured out why the claim is not "obvious", it is not hard to give a suitable argument (Exercise). □

Remark 2.2.12 (connection coefficients)**.** Let M be a smooth manifold of dimension n, let ∇ be a linear connection on M, let $\varphi\colon U \longrightarrow U'$ be a smooth chart for M, and let $(E_1, \dots, E_n)$ be the local frame of $\mathrm{T}\, M$ on U that is dual to $(\mathrm{d}\, \varphi^1, \dots, \mathrm{d}\, \varphi^n)$. For $i, j, k \in \{1, \dots, n\}$, we define the *connection coefficients* $({\Gamma_{ij}}^k)_{i,j.k \in \{1,\dots,n\}}$ of ∇ on U as the unique functions in $C^\infty(U)$ that satisfy

$$\forall_{i,j \in \{1,\dots,n\}} \quad \nabla^U_{E_i} E_j = \underbrace{{\Gamma_{ij}}^k E_k} = \sum_{k=1}^{n} {\Gamma_{ij}}^k \cdot E_k.$$

Then, we can express the restriction ∇^U as follows: If $X, Y \in \Gamma(\mathrm{T}\,U)$ and if $X = \sum_{i=1}^n X^i \cdot E_i$ and $Y = \sum_{j=1}^n Y^j \cdot E_j$ are their local descriptions, then

$$\begin{aligned}
\nabla^U_X Y &= \sum_{i=1}^n X^i \cdot \nabla^U_{E_i} Y && \text{(by (FL1))}\\
&= \sum_{i=1}^n X^i \cdot \sum_{j=1}^n \bigl(E_i(Y^j) \cdot E_j + Y^j \cdot \nabla^U_{E_i} E_j\bigr) && \text{(by (F2))}\\
&= \sum_{j=1}^n X(Y^j) \cdot E_j + \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n X^i \cdot Y^j\, \Gamma^k_{ij} \cdot E_k\\
&= \underbrace{X(Y^j)E_j}_{} + \underbrace{X^i Y^j \Gamma_{ij}{}^k E_k}_{}. && \text{(using Einstein summation)}
\end{aligned}$$

Conversely, if $(\Gamma_{ij}{}^k)_{i,j,k \in \{1,\dots,n\}}$ is a family in $C^\infty(U)$, then the preceding formula *defines* a linear connection on U (check!).

Example 2.2.13 (connection coefficients of the Euclidean connection)**.** The connection coefficients of the Euclidean connection $\overline{\nabla}$ on $\mathbb{R}^n$ with respect to the framing of $\mathrm{T}\,\mathbb{R}^n$ induced by the standard basis on $\mathbb{R}^n$ are ? .

With respect to polar coordinates, we obtain the following connection coefficients for the Euclidean connection $\overline{\nabla}$ on $\mathbb{R}^2 \setminus \{0\}$: We have – where we resort to fuzzy but highly intuitive notation –

$$\Gamma^r_{\vartheta\vartheta} = -r \quad \text{and} \quad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r} = \Gamma^\vartheta_{\vartheta r}$$

and all other connection coefficients are zero (Exercise)

2.2.3 Covariant derivatives along curves

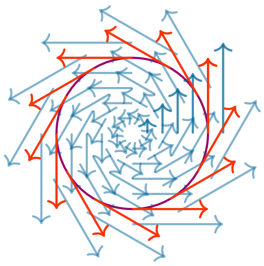
Connections allow us to differentiate vector fields along curves. In particular, this will enable us to define the acceleration of a smooth curve (with respect to a given connection) as the covariant derivative of the velocity vector field along the curve.

In order to formulate the defining properties of these covariant derivatives along curves, we need the following notions:

Definition 2.2.14 (extendable vector field)**.** Let M be a smooth manifold and let $\gamma\colon I \longrightarrow M$ be a smooth curve.

- We write $\Gamma(\mathrm{T}\,M|\gamma)$ for the set of all *vector fields in* $\mathrm{T}\,M$ *along* γ, i.e., for the set of all smooth maps $X\colon I \longrightarrow \mathrm{T}\,M$ that satisfy

$$\forall_{t \in I} \quad X(t) \in \mathrm{T}_{\gamma(t)}\,M.$$



Toggle vector field along the curve

Toggle extension vector field

Figure 2.2.: An extendable vector field, schematically

- A vector field $X \in \Gamma(\mathrm{T}\, M|\gamma)$ is *extendable*, if there exists a vector field $\widetilde{X} \in \Gamma(\mathrm{T}\, M)$ with

$$\forall_{t \in I} \quad X(t) = \widetilde{X}\bigl(\gamma(t)\bigr).$$

Quick check 2.2.15 ((non-)extendable vector fields)**.** We consider the smooth curve

$$\begin{aligned} \gamma \colon \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

in $\mathbb{R}^2$.

1. Is the vector field $t \longmapsto (-\sin t, \cos t)$ along γ extendable to $\mathbb{R}^2$?

   Yes No

2. Is the vector field $t \longmapsto (t, t)$ along γ extendable to $\mathbb{R}^2$?

   Yes No

Theorem 2.2.16 (covariant derivatives along curves)**.** *Let M be a smooth manifold, let ∇ be a linear connection on M, and let $\gamma \colon I \longrightarrow M$ be a smooth curve. Then there exists a unique map $D_\gamma \colon \Gamma(\mathrm{T}\, M|\gamma) \longrightarrow \Gamma(\mathrm{T}\, M|\gamma)$ with the following properties:*

- Linearity. *For all $\lambda_1, \lambda_2 \in \mathbb{R}$ and all $X_1, X_2 \in \Gamma(\mathrm{T}\, M|\gamma)$, we have*

$$D_\gamma(\lambda_1 \cdot X_1 + \lambda_2 \cdot X_2) = \lambda_1 \cdot D_\gamma(X_1) + \lambda_2 \cdot D_\gamma(X_2).$$

- Product rule. *For all $f \in C^\infty(I)$ and all $X \in \Gamma(\mathrm{T}\, M|\gamma)$, we have*

$$D_\gamma(f \cdot X) = f' \cdot X + f \cdot D_\gamma(X).$$

- Compatibility with the connection. *If $X \in \Gamma(\mathrm{T}\,M|\gamma)$ is extendable to $\widetilde{X}$, then for all $t \in I$:*

$$\big(D_\gamma(X)\big)(t) = \big(\nabla_{\dot\gamma(t)}(\widetilde{X})\big)\big(\gamma(t)\big).$$

This map D_γ is the covariant derivative along γ in M with respect to ∇.

Proof. We first show *uniqueness* by deriving suitable formulas from the three properties; in particular, this will also tell us how to prove existence.

Let $D_\gamma \colon \Gamma(\mathrm{T}\,M|\gamma) \longrightarrow \Gamma(\mathrm{T}\,M|\gamma)$ satisfy linearity, the product rule, and the compatiblity with the connection. Moreover, let $X \in \Gamma(\mathrm{T}\,M|\gamma)$, and let $t \in I$.

Similarly to the connection case (Proposition 2.2.9), one can show that $D_\gamma X(t)$ depends only on the values of X in an arbirarily small neighbourhood of t (check!).

We now derive a formula for $D_\gamma(X)(t)$: Let $\varphi \colon U \longrightarrow U'$ be a smooth chart for M around $\gamma(t)$ and let $(E_1, \dots, E_n)$ be the corresponding frame of $\mathrm{T}\,M$ over U. Then, on a small enough interval $J \subset I$ containing t, we can write

$$X = \sum_{j=1}^n X^j \cdot E_j \circ \gamma$$

for suitable smooth functions $X^1, \dots, X^n \colon J \longrightarrow \mathbb{R}$. Then the $E_1 \circ \gamma, \dots, E_n \circ \gamma$ are extendable on U by $E_1, \dots, E_n$. As $D_\gamma X(t)$ depends only on the values of X close to t, we can thus apply the compatibility property in this situation and obtain

$$\begin{aligned}
D_\gamma X(t) &= \sum_{j=1}^n D_\gamma(X^j \cdot E_j \circ \gamma)(t) && \text{(linearity)}\\
&= \sum_{j=1}^n X^{j\prime}(t) \cdot E_j \circ \gamma(t) + \sum_{j=1}^n X^j(t) \cdot D_\gamma(E_j \circ \gamma)(t) && \text{(product rule)}\\
&= \sum_{j=1}^n X^{j\prime}(t) \cdot E_j \circ \gamma(t) + \sum_{j=1}^n X^j(t) \cdot \nabla^U_{\gamma(t)}(E_j)(\gamma(t)). && \text{(compatibility)}
\end{aligned}$$

In particular, if $(\Gamma^k_{ij})_{i,j,k}$ are the connection coefficients of ∇ in these local coordinates, then we obtain

$$\begin{aligned}
D_\gamma X(t) = {} & \sum_{j=1}^n X^{j\prime}(t) \cdot E_j \circ \gamma(t)\\
& + \sum_{k=1}^n \sum_{i=1}^n \sum_{j=1}^n \gamma^{i\prime}(t) \cdot X^j(t) \cdot \Gamma^k_{ij}(\gamma(t)) \cdot E_k \circ \gamma(t).
\end{aligned}$$

As the right-hand side is phrased completely in terms of the original connection ∇ and the given curve γ, we completed the proof of uniqueness.

We now show *existence*: Locally, we use the local coordinate description above as a *definition*. A straightforward computation shows that this construction satisfies linearity, the product rule, and the compatibility condition (check!). Because of uniqueness, these local definitions match on the intersection of smooth charts and thus lead to a well-defined map $D_\gamma \colon \Gamma(\mathrm{T}\,M|\gamma) \longrightarrow \Gamma(\mathrm{T}\,M|\gamma)$ (check!), which again satisfies linearity, the product rule, and the compatibility condition (check!). $\square$

We give two prototypical examples of covariant derivatives along curves: In the first one, we use local coordinates; in the second, we refer more directly to the axioms.

Example 2.2.17 (covariant derivative along curves, Euclidean case)**.** Let $n \in \mathbb{N}$ and let $\gamma \colon I \longrightarrow \mathbb{R}^n$ be a smooth curve in $\mathbb{R}^n$. Then the local description of the covariant derivative D_γ with respect to the Euclidean linear connection on $\mathbb{R}^n$ shows that (because the connection coefficients are zero in the standard coordinates on $\mathbb{R}^n$; Example 2.2.13) that

$$D_\gamma \dot{\gamma} = \gamma'',$$

where $\gamma'' \colon I \longrightarrow \mathbb{R}^n$ is the usual second derivative of γ (check!). In particular:

- If $\gamma \colon I \longrightarrow \mathbb{R}^n$ is a straight line segment, i.e., if there exist $a, b \in \mathbb{R}^n$ with $\gamma = (t \mapsto a + t \cdot (b - a))$, then

$$D_\gamma \dot{\gamma} = \boxed{?} \qquad .$$

- If $\gamma \colon I \longrightarrow \mathbb{R}^n$ is given by $(t \mapsto (\cos t, \sin t))$, then

$$\forall_{t \in I} \quad D_\gamma \dot{\gamma}(t) = \boxed{?} \qquad\qquad\qquad .$$

Example 2.2.18 (covariant derivative along curves, on the circle)**.** We consider the linear connection $\nabla^\top$ on $\mathbb{S}^1$ induced by the Euclidean linear connection on $\mathbb{R}^2$ (Example 2.2.8) and the smooth curve

$$\begin{aligned} \gamma \colon \mathbb{R} &\longrightarrow \mathbb{S}^1 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

on $\mathbb{S}^1$. Moreover, let $X := \dot{\gamma} \in \Gamma(\mathrm{T}\,\mathbb{S}^1|\gamma)$ be the velocity vector field along γ.

We now compute $D_\gamma(X)$ with respect to $\nabla^\top$: Both D_γ and $\nabla^\top$ are defined in terms of suitable extensions of the vector field X. In fact,

$$\begin{aligned} \widetilde{X} \colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto \boxed{?} \end{aligned}$$

is an extension of X to $\mathbb{R}^2$ (using the canonical identification of the tangent spaces of $\mathbb{R}^2$ with $\mathbb{R}^2$; check!).

Let E_1, E_2 be the standard coordinate frame for $\operatorname{T}\mathbb{R}^2$ and let $p\colon \operatorname{T}\mathbb{R}^2 \longrightarrow \operatorname{T}\mathbb{S}^1$ denote the orthogonal projection. Then, for all $t \in \mathbb{R}$, we obtain:

$$\begin{aligned}
D_\gamma(X)(t) &= \bigl(\nabla^\top_X(\widetilde{X}|_{\mathbb{S}^1})\bigr)(\gamma(t)) \\
&= p_{\gamma(t)}\bigl((\overline{\nabla}_{\widetilde{X}}\widetilde{X})(\gamma(t))\bigr) \\
&= p_{\gamma(t)}\bigl(-\sin t \cdot (\overline{\nabla}_{E_1}\widetilde{X})(\gamma(t)) + \cos t \cdot (\overline{\nabla}_{E_2}\widetilde{X})(\gamma(t))\bigr) \\
&= p_{\gamma(t)}\bigl(-\sin t \cdot (0,1) + \cos t \cdot (-1,0)\bigr) \\
&= p_{\gamma(t)}(-\cos t, -\sin t) \\
&= \boxed{?} \quad .
\end{aligned}$$

2.2.4 Geodesics

On a smooth manifold with a linear connection, the acceleration of a smooth curve γ is the covariant derivative $D_\gamma\dot\gamma$ of the velocity along the given curve. In Euclidean space, a smooth curve is a line (segment) if and only if its acceleration is zero (check!). In the setting of manifolds, we turn this property into the definition of being geodesic.

The geodesic condition locally can be reformulated in terms of an ordinary differential equation. Therefore, basics on ordinary differential equations lead to local existence and uniqueness statements for geodesics.

Definition 2.2.19 (geodesic)**.** Let M be a smooth manifold and let ∇ be a linear connection on M. A smooth curve $\gamma\colon I \longrightarrow M$ is a *geodesic (with respect to* ∇*)*, if

$$D_\gamma\dot\gamma = 0.$$

Here, D_γ is the covariant derivative along γ with respect to ∇.

Example 2.2.20 (geodesics)**.**

- Let $n \in \mathbb{N}$. Then, for all $a, b \in \mathbb{R}^n$, the straight line

$$\begin{aligned}
\mathbb{R} &\longrightarrow \mathbb{R}^n \\
t &\longmapsto a + t \cdot (b - a)
\end{aligned}$$

  is a geodesic in $\mathbb{R}^n$ with respect to the Euclidean linear connection (Example 2.2.17). Conversely, every geodesic in $\mathbb{R}^n$ is a straight (affine) line (check!)

- The smooth curve

$$\begin{aligned}
\mathbb{R} &\longrightarrow \mathbb{R}^2 \\
t &\longmapsto (\cos t, \sin t)
\end{aligned}$$

is *not* a geodesic in $\mathbb{R}^2$ with respect to the Euclidean linear connection (Example 2.2.17).

- The smooth curve

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{S}^1 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

is a geodesic in $\mathbb{S}^1$ with respect to the linear connection on $\mathbb{S}^1$ induced by the Euclidean linear connection on $\mathbb{R}^2$ (Example 2.2.18).

In particular, in general, geodesics need *not* be injective!

Remark 2.2.21 (the geodesic equation)**.** The local description of the covariant derivative along curves (proof of Theorem 2.2.16) leads to the following local characterisation of geodesics:

Let M be a smooth manifold, let ∇ be a linear connection on M, let $\gamma \colon I \longrightarrow M$ be a smooth curve, let $t \in I$, and let $\varphi \colon U \longrightarrow U'$ be a smooth chart for M around $\gamma(t)$. Let $(\Gamma^k_{ij})_{i,j,k}$ denote the connection coefficients of ∇ with respect to the frame for $\operatorname{T} M$ over U induced by φ.

Moreover, let $J \subset I$ be an interval with $t \in J$ and $\gamma(J) \subset U$ and let $(\gamma^1, \dots, \gamma^n)$ be the coordinate functions of γ on J with respect to φ. Then $(D_\gamma \dot\gamma)|_J = 0$ if and only if

$$\forall_{k \in \{1,\dots,n\}} \quad \forall_{s \in J} \quad {\gamma^k}''(s) + \sum_{i=1}^n \sum_{j=1}^n {\gamma^i}'(s) \cdot {\gamma^j}'(s) \cdot \Gamma^k_{ij}(\gamma(s)) = 0$$

(check!). This ordinary differential equation is thus called the *geodesic equation*.

Theorem 2.2.22 (existence and uniqueness of geodesics)**.** *Let M be a smooth manifold, let ∇ be a linear connection on M, let $x \in M$, let $v \in \operatorname{T}_x M$, and let $t \in \mathbb{R}$. Then, there exists an open interval $I \subset \mathbb{R}$ with $t \in I$ and a geodesic $\gamma \colon I \longrightarrow M$ (with respect to ∇) with*

$$\gamma(t) = x \quad \textit{and} \quad \dot\gamma(t) = v.$$

Moreover, if $\eta \colon J \longrightarrow M$ is also a geodesic with $t \in J$ and $\eta(t) = x$ and $\dot\eta(t) = v$, then $\gamma|_{I \cap J} = \eta|_{I \cap J}$.

Proof. The property of being a geodesic is characterised by a second order ordinary differential equation (Remark 2.2.21). The conditions "$\gamma(t) = x$" and "$\dot\gamma(t) = v$" provide initial values for the functions and their first derivatives (check!). Therefore, the classical existence and uniqueness theorems for ordinary differential equations (with smooth equation coefficients) apply and prove the claimed local existence and uniqueness of geodesics (check!). $\square$

Definition 2.2.23 (maximal geodesic)**.** Let M be a smooth manifold with a linear connection ∇. A smooth curve $\gamma\colon I \longrightarrow M$ is a *maximal geodesic (with respect to* ∇*)*, if γ is a geodesic with respect to ∇ and if there is no geodesic $\eta\colon J \longrightarrow M$, where J is an interval with $I \subsetneq J$ and $\eta|_I = \gamma$.

Corollary 2.2.24 (existence and uniqueness of maximal geodesics)**.** *Let M be a smooth manifold, let ∇ be a linear connection on M, let $x \in M$, and let $v \in \mathrm{T}_x M$. Then there is a unique maximal geodesic $\gamma\colon I \longrightarrow M$ with $0 \in I^\circ$ and*

$$\gamma(0) = x \quad \textit{and} \quad \dot\gamma(0) = v.$$

We will denote this maximal geodesic by $\operatorname{geod}_{x,v}$.

Proof. This follows via standard arguments on maximal solutions for ordinary differential equations from Theorem 2.2.22 (Exercise). □

Example 2.2.25 (maximal geodesics)**.**

- Let $n \in \mathbb{N}$. Then the maximal geodesics in $\mathbb{R}^n$ are nothing but all linearly parametrised affine lines (Example 2.2.20; check!).
- We now consider the punctured plane $\mathbb{R}^2 \setminus \{0\}$ with the connection induced from the Euclidean linear connection on $\mathbb{R}^2$ via Propsition 2.2.11.

  Then, for $x = (-1, 0)$ and $v = (1, 0)$, we obain that $\operatorname{geod}_{x,v}$ has the domain Hint and is given by (check!)

  $$\begin{aligned} \operatorname{geod}_{x,v}\colon (-\infty, 1) &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ t &\longmapsto x + t \cdot (1, 0). \end{aligned}$$

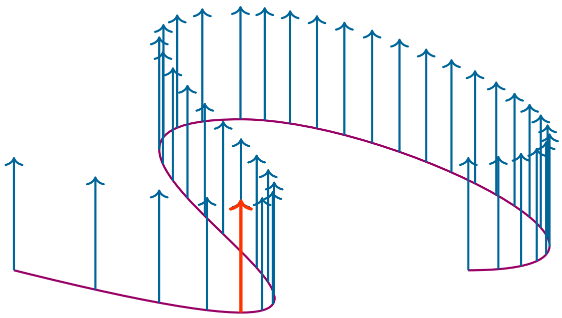
  In particular, the domain of this geodesic is *not* all of $\mathbb{R}$.

In order to relate this notion of geodesic to actual geometric properties, we need to let the connection interact in the appropriate way with an underlying Riemannian metric. We will do this in Chapter 2.3.

2.2.5 Parallel transport

Vector fields along curves whose covariant derivative is zero are considered parallel. We will see that every tangent vector can be extended to a unique parallel vector field along the given curve. In particular, this allows to move between tangent spaces along a curve, leading to parallel transport along the curve (Figure 2.3).

In this sense, connections "connect" different tangent spaces. It turns out that we can actually recover the covariant derivative along curves from parallel transport and thus that we can recover linear connections from their parallel transport. This gives a nice geometric interpretation of connections.



Toggle initial vector

Toggle parallel transport

Figure 2.3.: Parallel transport along a curve, schematically

Definition 2.2.26 (parallel vector field)**.** Let M be a smooth manifold with a linear connection ∇, let $\gamma\colon I \longrightarrow M$ be a smooth curve, and let $X \in \Gamma(\operatorname{T} M|\gamma)$. Then X is *parallel along* γ, if

$$D_\gamma X = 0,$$

where D_γ is the covariant derivative of X along γ.

Quick check 2.2.27. How can the notion of geodesics be reformulated in terms of parallel vector fields?

Hint

Remark 2.2.28 (parallel transport equation)**.** As in the case of geodesics, the local description of the covariant derivative along curves (proof of Theorem 2.2.16) leads to a local characterisation of parallel vector fields:

Let M be a smooth manifold, let ∇ be a linear connection on M, let $\gamma\colon I \longrightarrow M$ be a smooth curve, let $t \in I$, and let $\varphi\colon U \longrightarrow U'$ be a smooth chart for M around $\gamma(t)$. Let $(\Gamma^k_{ij})_{i,j,k}$ denote the connection coefficients of ∇ with respect to the frame induced by φ.

Moreover, let $J \subset I$ be an open interval with $t \in J$ and $\gamma(J) \subset U$, let $X^1, \dots, X^n$ denote the coordinate functions of X on J with respect to φ. Then $(D_\gamma X)|_J = 0$ if and only if

$$\forall_{k \in \{1,\dots,n\}} \quad \forall_{s \in J} \quad X^{k\prime}(s) = -\sum_{j=1}^n \sum_{i=1}^n X^j(s) \cdot \gamma^{i\prime}(s) \cdot \Gamma^k_{ij}\bigl(\gamma(s)\bigr)$$

(check!). It should be noted that this is a linear ordinary differential equation with smooth coefficients.

Example 2.2.29 (parallel vector field in Euclidean space)**.** Let $n \in \mathbb{N}$ and let $\gamma\colon I \longrightarrow \mathbb{R}^n$ be a smooth curve. Then $X \in \Gamma(\mathrm{T}\,\mathbb{R}^n|\gamma)$ is parallel with respect to the Euclidean linear connection if and only if X is constant (viewed as map $I \longrightarrow \mathbb{R}^n$), because the connection coefficients with respect to the standard coordinates are zero (check!).

Theorem 2.2.30 (existence and uniqueness of parallel vector fields)**.** *Let M be a smooth manifold with a linear connection ∇, let $\gamma\colon I \longrightarrow M$ be a smooth curve, let $t \in I$, and let $v \in \mathrm{T}_{\gamma(t)}\,M$. Then there exists a unique $X \in \Gamma(\mathrm{T}\,M|\gamma)$ that is parallel along γ and satisfies $X(t) = v$.*

Proof. We use the characterisation in terms of the parallel transport equation (Remark 2.2.28), which is a linear ordinary differential equation with smooth coefficients.

- *Local step.* If γ is contained in a single chart, then the usual existence and uniqueness theorem for linear ordinary differential equations shows that we can find a unique solution to the parallel transport equation along *all* of γ with given initial value v at t (check!).

- *Global step.* We apply the standard extension argument for constructing global solutions:

  Let B be the supremum of all $b \in I$ with $b > t$ such that there exists an $X \in \Gamma(\mathrm{T}\,M|\gamma|_{[t,b]})$ that is parallel along $\gamma|_{[t,b]}$ and that satisfies $X(t) = v$. It now suffices to show that $B = \sup I$.

  Assume for a contradiction that $B \neq \sup I$. Then we can apply the local step to γ around B: Let U be a chart domain around $\gamma(B)$, let $\varepsilon \in \mathbb{R}_{>0}$ with $J := [B-\varepsilon, B+\varepsilon] \subset I$ and $\gamma(J) \subset U$, and let X be parallel along γ on $[t, B-\varepsilon/2]$ with $X(t) = v$. Then the local step shows that there is a parallel vector field Y on J along $\gamma|_J$ with $Y(B-\varepsilon/2) = X(B-\varepsilon/2)$. By uniqueness, $Y|_{[B-\varepsilon,B-\varepsilon/2]} = X|_{[B-\varepsilon,B-\varepsilon/2]}$, which contradicts the defining property of B. Hence, $B = \sup I$. ☐

Definition 2.2.31 (parallel transport)**.** Let M be a smooth manifold with a linear connection ∇, let $\gamma\colon I \longrightarrow M$ be a smooth curve, and let $s, t \in I$. Then the *parallel transport along γ from s to t* is defined as

$$\begin{aligned} P^\gamma_{s,t}\colon \mathrm{T}_{\gamma(s)}\,M &\longrightarrow \mathrm{T}_{\gamma(t)}\,M \\ v &\longmapsto X(t), \text{ where } X \in \Gamma(\mathrm{T}\,M|\gamma) \text{ is parallel along } \gamma \text{ and } X(s) = v. \end{aligned}$$

Quick check 2.2.32 (on the definition of parallel transport)**.** Why is $P^\gamma_{s,t}$ in Definition 2.2.31 well-defined?

Because

Proposition 2.2.33. *Let M be a smooth manifold with a linear connection ∇ and let $\gamma\colon I \longrightarrow M$ be a smooth curve. Then, for all $s, t \in I$, parallel transport*

$$P^{\gamma}_{s,t} \colon \mathrm{T}_{\gamma(s)} M \longrightarrow \mathrm{T}_{\gamma(t)} M$$

along γ is an $\mathbb{R}$-linear isomorphism.

Proof. As parallel transport is characterised in terms of a linear ordinary differential equation (Remark 2.2.28), with the input tangent vector being the initial condition, we obtain that $P^{\gamma}_{s,t}$ is $\mathbb{R}$-linear (check!).

Moreover, $P^{\gamma}_{s,t}$ is an isomorphism, because which map? is an inverse (check!). $\square$

Theorem 2.2.34 (covariant derivatives along curves via parallel transport)**.** *Let M be a smooth manifold with a linear connection ∇, let $\gamma \colon I \longrightarrow M$ be a smooth curve, let $X \in \Gamma(\mathrm{T}\, M|\gamma)$, and let $t \in I$. Then:*

$$(D_\gamma X)(t) = \lim_{s \to t} \frac{P^{\gamma}_{s,t} X(s) - X(t)}{s - t}.$$

Proof. The idea is to express both sides of the claimed equation in terms of parallel frames: Let n be the dimension of M.

A *parallel frame* along γ is a family $(E_1, \dots, E_n)$ in $\Gamma(\mathrm{T}\, M|\gamma)$ with the property that

- for each $s \in I$, the family $(E_1(s), \dots, E_n(s))$ is a basis of $\mathrm{T}_{\gamma(s)} M$ and
- such that $E_1, \dots, E_n$ are parallel along γ.

In view of the existence of parallel vector fields along γ (Theorem 2.2.30), we know that parallel frames along γ exist. How?

As $(E_1, \dots, E_n)$ is a smooth frame for $\mathrm{T}\, M$ along γ, there exist smooth functions $X^1, \dots, X^n \colon I \longrightarrow \mathbb{R}$ with

$$\forall_{s \in I} \quad X(s) = \sum_{i=1}^{n} X^i(s) \cdot E_i(s).$$

We now rewrite both sides of the claimed equation:

- *Left-hand side.* We have

$$\begin{aligned}
D_\gamma X &= \sum_{i=1}^{n} D_\gamma (X^i \cdot E_i) && \text{(linearity)} \\
&= \sum_{i=1}^{n} (X^{i\prime} \cdot E_i + X^i \cdot D_\gamma E_i) && \text{(product rule)} \\
&= \sum_{i=1}^{n} X^{i\prime} \cdot E_i. && \text{(Why?)}
\end{aligned}$$

- *Right-hand side.* For all $s \in I$, we have

$$
\begin{aligned}
P^{\gamma}_{s,t}\bigl(X(s)\bigr) &= \sum_{i=1}^{n} X^i(s) \cdot P^{\gamma}_{s,t}(E_i(s)) && (P^{\gamma}_{s,t} \text{ is linear}) \\
&= \sum_{i=1}^{n} X^i(s) \cdot E_i(t). && \text{(parallelism)}
\end{aligned}
$$

Therefore, the differential quotient on the right-hand side coincides with the left-hand side of the claimed equation. □

Corollary 2.2.35 (connections via parallel transport)**.** *Let M be a smooth manifold with a linear connection ∇, let $X, Y \in \Gamma(\mathrm{T}\, M)$, and let $x \in M$. If $\gamma\colon I \longrightarrow M$ is a smooth curve in M with $0 \in I$ and*

$$\gamma(0) = x \quad \textit{and} \quad \dot\gamma(0) = X(x),$$

then

$$(\nabla_X Y)(x) = \lim_{h \to 0} \frac{P^{\gamma}_{h,0}(Y(\gamma(h)) - Y(x)}{h}.$$

Proof. We first express the connection in terms of covariant derivatives along curves; then, we use the fact that covariant derivatives can be expressed in terms of parallel transport (Theorem 2.2.34; Exercise). □

2.3 The Levi-Civita connection

We will now add the Riemannian metric to the picture. In fact, we will add two further conditions to connections:

- The first one encodes compatiblity with a given Riemannian metric. Geometrically, this means that the parallel transport maps defined by the connection are all isometric with respect to the Riemannian metric.
- The second condition is a symmetry condition that can be formulated as symmetry of the connection coefficients, which relates swapped covariant derivatives with the Lie bracket.

In order to formulate and charaterise the compatibility and symmetry conditions for connections, it is convenient to extend linear connections to the tensor bundles of the tangent bundle. We will do this in a first, preparatory, step.

The fundamental theorem of Riemannian geometry then states that on every Riemannian manifold there exists a unique linear connection that satisfies compatibility and symmetry. This is the so-called *Levi-Civita connection.*

We will see in the next chapter that the Levi-Civita connection accurately reflects the metric properties of geodesics: Geodesics with respect to the Levi-Civita connection are locally length-minimising (Chapter 3.2).

Moreover, we will reach our first intermediate goal: The non-commutativity of iterated covariant derivatives with respect to the Levi-Civita connection can be exploited to give an analytic definition of various curvature tensors (Chapter 2.4).

2.3.1 Connections on tensor bundles

Linear connections induce corresponding connections on all tensor bundles. Constructions of this type are vastly simplified by the tensor characterisation lemma (Proposition 2.3.2), which describes tensor fields in terms of their actions on sections. We will mainly consider only the two following types of induced connections, which will lead to a uniform notation later on:

Remark 2.3.1 (connections on functions and on parametrised bilinear forms)**.** Let M be a smooth manifold and let ∇ be a linear connection on M.

- We use the notation (even though this is independent of the given ∇)

$$\begin{aligned} \nabla\colon \Gamma(\mathrm{T}\, M) \times C^\infty(M) &\longrightarrow C^\infty(M) \\ (X, f) &\longmapsto X(f) \end{aligned}$$

  for the induced connection on the trivial tensor bundle $\mathbf{T}^{0,0}\, M \cong M \times \mathbb{R}$, whose sections are canonically isomorphic to $C^\infty(M)$ (Remark 1.2.27). In fact, this defines a connection on $\mathbf{T}^{0,0}\, M$ (check!).

- We set

$$\begin{aligned} \nabla\colon \Gamma(\mathrm{T}\, M) \times \Gamma(\mathbf{T}^{2,0}\, M) &\longrightarrow \Gamma(\mathbf{T}^{2,0}\, M) \\ (X, g) &\longmapsto \mathrm{TC}^{-1}\bigl((Y,Z) \mapsto X(\langle Y, Z\rangle_g) - \langle \nabla_X Y, Z\rangle_g - \langle Y, \nabla_X Z\rangle_g\bigr), \end{aligned}$$

  where we used the notation of the tensor characterisation lemma (Proposition 2.3.2) and replaced terms of the form "$g(Y \otimes Z)$" by the more suggestive version "$\langle Y, Z\rangle_g$".

  This defines a connection on $\mathbf{T}^{2,0}(M)$ (check!).

Proposition 2.3.2 (tensor characterisation lemma)**.** *Let M be a smooth manifold and let $k, \ell \in \mathbb{N}$.*

1. *If $g \in \Gamma(\mathbf{T}^{k,\ell}\, M)$, then*

$$\begin{aligned} \mathrm{TC}(g)\colon \Gamma(\mathrm{T}\, M)^{\times k} \times \Gamma(\mathrm{T}^*\, M)^{\times \ell} &\longrightarrow C^\infty(M) \\ (X_1, \dots, X_k, Y_1, \dots, Y_\ell) &\longmapsto g(X_1 \otimes \dots \otimes X_k \otimes Y_1 \otimes \dots \otimes Y_\ell) \\ &:= \bigl(x \mapsto g_x(X_1(x) \otimes \dots \otimes X_k(x) \otimes Y_1(x) \otimes \dots \otimes Y_\ell(x))\bigr) \end{aligned}$$

is a well-defined $C^\infty(M)$-multilinear map.

2. *The map*

$$\mathrm{TC}\colon \Gamma(\mathbf{T}^{k,\ell} M) \longrightarrow \textit{(set of } C^\infty(M)\textit{-multilinear maps } \Gamma(\mathrm{T}\, M)^k \times \Gamma(\mathrm{T}^*\, M)^\ell \to C^\infty(M)\textit{)}$$

is bijective.

Proof. Ad 1. This is a straightforward computation (Exercise).

Ad 2. The key point is to show surjectivity. Maps that satisfy the $C^\infty(M)$-multilinearity condition are "local", similarly to the locality of connections (Proposition 2.2.9). Therefore, one can work locally to show the existence of a preimage (Exercise). □

Caveat 2.3.3. It is common to not make the application of TC or its inverse explicit in the notation. For example, one usually just says that "$(Y,Z) \mapsto X(\langle Y,Z\rangle_g) - \langle \nabla X_Y, Z\rangle_g - \langle Y, \nabla_X Z\rangle_g$" is a $(2,0)$-tensor field.

Remark 2.3.4 (tensor characterisation lemma, vector field values)**.** Moreover, there is the following version with an integrated (un)currying step: Let M be a smooth manifold and let $k,\ell \in \mathbb{N}$. In view of Remark 1.2.27, we can canonically identify $\mathbf{T}^{k,\ell+1} M$ with the smooth vector bundle

$$\mathrm{Hom}_{\mathbb{R}}\bigl((\mathrm{T}\, M)^{\otimes k} \otimes_{\mathbb{R}} (\mathrm{T}^*\, M)^{\otimes \ell}, \mathrm{T}\, M\bigr).$$

Therefore, every $g \in \Gamma(\mathbf{T}^{k,\ell+1} M)$ leads to a $C^\infty(M)$-multilinear map

$$\Gamma(\mathrm{T}\, M)^{\times k} \times \Gamma(\mathrm{T}^*\, M)^{\times \ell} \longrightarrow \Gamma(\mathrm{T}\, M).$$

Similarly to the tensor characterisation lemma (Proposition 2.3.2), this also leads to a bijective correspondence and we can construct or describe elements of $\Gamma(\mathbf{T}^{k,\ell+1} M)$ via their associated evaluation maps (check!).

More generally, linear connections also induce connections on all higher tensor bundles in a canonical way [18, Proposition 4.15].

2.3.2 The Levi-Civita connection

We first formalise the two geometrically relevant additional conditions on connections:

- The compatibility with the underlying Riemannian metric and
- the symmetry condition.

In addition to the formal definitions, we will also give alternative characterisations that indicate the geometric and analytic relevance of these constraints.

We will then prove the fundamental theorem of Riemannian geometry and give basic first examples.

We begin with compatibility with the Riemannian metric:

Definition 2.3.5 (compatible connection)**.** Let (M, g) be a Riemannian manifold. A linear connection ∇ on M is *compatible with* g if the following holds: For all $X, Y, Z \in \Gamma(\mathrm{T}\, M)$, we have

$$\nabla_X \langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g.$$

Quick check 2.3.6 (on the definition of compatibiliy)**.** In Definition 2.3.5:

1. What is "∇" in the defining equality?

   Hint

2. What is a good way to remember the compatibility condition equation?

   Suggestion

Proposition 2.3.7 (compatible connections, geometric characterisation)**.** *Let (M, g) be a Riemannian manifold and let ∇ be a linear connection on M. Then the following are equivalent:*

1. *The connection ∇ is compatible with the Riemannian metric g.*
2. *The Riemannian metric g is parallel to ∇, i.e.: For all $X \in \Gamma(\mathrm{T}\, M)$, we have $\nabla_X g = 0$.*
3. *For all smooth curves γ on M and all $X, Y \in \Gamma(\mathrm{T}\, M|\gamma)$, we have*

$$\langle X, Y\rangle_g' = \langle D_\gamma X, Y\rangle_g + \langle X, D_\gamma Y\rangle_g.$$

4. *For all smooth curves $\gamma \colon I \longrightarrow M$ and all $s, t \in I$, the parallel transport $P^\gamma_{s,t} \colon \mathrm{T}_{\gamma(s)}\, M \longrightarrow \mathrm{T}_{\gamma(t)}\, M$ is an $\mathbb{R}$-linear isometry with respect to the norms on the tangent spaces induced by g.*

Proof. Ad 1. $\Longleftrightarrow$ 2. This is a direct consequence of the definition of compatibility and the definition of $\nabla_X g$ (Remark 2.3.1).

Ad 1. $\Longrightarrow$ 3. As this is about translating the compatibility condition for ∇ into the corresponding condition for the induced covariant derivatives along curves, we will argue via suitable extendable vector fields. Let ∇ be compatible with g, let γ be a smooth curve on M, and let $X, Y \in \Gamma(\mathrm{T}\, M|\gamma)$. Both sides of 3. are local, thus we may work in a single chart (check!). Let $t \in I$, let $\varphi \colon U \longrightarrow U'$ be a smooth chart around $\gamma(t)$, and let $(E_1, \dots, E_n)$ be the frame of $\mathrm{T}\, M$ over U associated to φ. Then, we can write $X = \sum_{i=1}^n X^i \cdot E_i \circ \gamma$ and $Y = \sum_{j=1}^n Y^j \cdot E_j \circ \gamma$ in a neighbourhood J of t, with smooth functions $X^1, \dots, X^n, Y^1, \dots, Y^n \colon J \longrightarrow \mathbb{R}$. On J, we obtain (phew!)

$$
\begin{aligned}
\langle X, Y\rangle_g' &= \Big\langle \sum_{i=1}^n X^i \cdot E_i \circ \gamma, \sum_{j=1}^n Y^j \cdot E_j \circ \gamma \Big\rangle_g' \\
&= \sum_{i=1}^n \sum_{j=1}^n (X^{i\prime} \cdot Y^j + X^i \cdot Y^{j\prime}) \cdot \langle E_i \circ \gamma, E_j \circ \gamma\rangle_g \\
&+ \sum_{i=1}^n \sum_{j=1}^n X^i \cdot Y^j \cdot \langle E_i \circ \gamma, E_j \circ \gamma\rangle_g' && \text{(by the product rule)} \\
&= \sum_{i=1}^n \sum_{j=1}^n (X^{i\prime} \cdot Y^j + X^i \cdot Y^{j\prime}) \cdot \langle E_i \circ \gamma, E_j \circ \gamma\rangle_g \\
&+ \sum_{i=1}^n \sum_{j=1}^n X^i \cdot Y^j \cdot \dot\gamma\big(\langle E_i, E_j\rangle_g\big) && \text{(viewing } \dot\gamma \text{ at each point as derivation)} \\
&= \sum_{i=1}^n \sum_{j=1}^n (X^{i\prime} \cdot Y^j + X^i \cdot Y^{j\prime}) \cdot \langle E_i \circ \gamma, E_j \circ \gamma\rangle_g \\
&+ \sum_{i=1}^n \sum_{j=1}^n X^i \cdot Y^j \cdot \big(\langle \nabla_{\dot\gamma} E_i, E_j\rangle_g + \langle E_i, \nabla_{\dot\gamma} E_j\rangle_g\big) && (\text{Why?}) \\
&= \sum_{i=1}^n \sum_{j=1}^n (X^{i\prime} \cdot Y^j + X^i \cdot Y^{j\prime}) \cdot \langle E_i \circ \gamma, E_j \circ \gamma\rangle_g \\
&+ \sum_{i=1}^n \sum_{j=1}^n X^i \cdot Y^j \cdot \big(\langle D_\gamma(E_i \circ \gamma), E_j \circ \gamma\rangle_g + \langle E_i \circ \gamma, D_\gamma(E_j \circ \gamma)\rangle_g\big) && \text{(definition of } D_\gamma) \\
&= \langle D_\gamma X, Y\rangle_g + \langle X, D_\gamma Y\rangle_g && \text{(defining properties of } D_\gamma\text{; check!)}
\end{aligned}
$$

Ad 3. $\Longrightarrow$ 1. Conversely, suppose that 3. holds and let $X, Y, Z \in \Gamma(\mathrm{T}\, M)$. We express the connection in terms of the covariant derivative along curves: Let $x \in M$ and let γ be a smooth curve with $\dot\gamma(0) = X(x)$. Then $X \circ \gamma, Y \circ \gamma \in \Gamma(\mathrm{T}\, M|\gamma)$ and

$$
\begin{aligned}
\nabla_X \langle Y, Z\rangle_g(x) &= X(x)(\langle Y, Z\rangle_g) = \dot\gamma(0)(\langle Y, Z\rangle_g) = \langle Y \circ \gamma, Z \circ \gamma\rangle_g'(0) \\
&= \big\langle D_\gamma(Y \circ \gamma), Z\big\rangle_g(0) + \big\langle Y, D_\gamma(Z \circ \gamma)\big\rangle_g(0) && \text{(by 3.)} \\
&= \langle \nabla_{\dot\gamma} Y, Z\rangle_g(0) + \langle Y, \nabla_{\dot\gamma} Z\rangle_g(0) && \text{(definition of } D_\gamma) \\
&= \langle \nabla_X Y, Z\rangle_g(x) + \langle Y, \nabla_X Z\rangle_g(x). && \text{(locality of } \nabla)
\end{aligned}
$$

Ad 3. $\Longrightarrow$ 4. As $P^\gamma_{s,t}$ is an $\mathbb{R}$-linear isomorphism (Proposition 2.2.33), we only need to establish compatibility with the inner products. To this end, we unravel the definition of parallel transport: Let $v, w \in \mathrm{T}_{\gamma(s)} M$. Then there exist parallel $X, Y \in \Gamma(\mathrm{T}\, M|\gamma)$ with $X(s) = v$ and $Y(s) = w$ (Theorem 2.2.30)). Because X and Y are parallel, 3. implies that

$$
\langle X, Y\rangle_g' = \langle D_\gamma X, Y\rangle + \langle X, D_\gamma Y\rangle_g = \boxed{?}
$$

and thus that $\langle X, Y\rangle_g$ is constant. In particular, we obtain

$$
\begin{aligned}
\big\langle P^\gamma_{s,t}(v), P^\gamma_{s,t}(w)\big\rangle_{g(t)} &= \big\langle X(t), Y(t)\big\rangle_{g(t)} && \text{(by definition of } P^\gamma_{s,t}) \\
&= \big\langle X(s), Y(s)\big\rangle_{g(s)} && \text{(because } \langle X, Y\rangle_g \text{ is constant)} \\
&= \langle v, w\rangle_{g(s)}.
\end{aligned}
$$

This shows that $P^\gamma_{s,t}$ is an isometry.

Ad 4. $\Longrightarrow$ 3. One can use parallel orthonormal frames to deduce 3. from 4. (Exercise). □

Proposition 2.3.8 (compatible connections, in coordinates)**.** *Let (M, g) be a Riemannian manifold of dimension n and let ∇ be a linear connection on M. Then the following are equivalent:*

1. *The connection ∇ is compatible with the Riemannian metric g.*
2. *For all coordinate frames $(E_i)_{i\in\{1,\dots,n\}}$ on a smooth chart domain U, we have*

$$\forall_{i,j,k\in\{1,\dots,n\}} \quad \nabla^U_{E_i}\langle E_j, E_k\rangle_g = \langle \nabla^U_{E_i} E_j, E_k\rangle_g + \langle E_j, \nabla^U_{E_i} E_k\rangle_g.$$

3. *For all coordinate frames $(E_i)_{i\in\{1,\dots,n\}}$ on a smooth chart domain, we have*

$$\forall_{i,j,k\in\{1,\dots,n\}} \quad \sum_{\ell=1}^n ({\Gamma_{ki}}^\ell\cdot g_{\ell j} + {\Gamma_{kj}}^\ell \cdot g_{i\ell}) = E_k(g_{ij}),$$

where $({\Gamma_{ij}}^k)_{i,j,k\in\{1,\dots,n\}}$ denote the connection coefficients with respect to this coordinate frame and where $(g_{ij})_{i,j\in\{1,\dots,n\}}$ denote the coordinates of g with respect to the underlying chart.

Proof. *Ad* 1. $\Longleftrightarrow$ 2. If 1. is satisfied, then certainly 2. is satisfied. Conversely, if 2. is satisfied, then rewriting general vector fields in terms of the coordinate frame $(E_i)_{i\in\{1,\dots,n\}}$ and the product rules show that ∇ is compatible with g (check!).

Ad 2. $\Longleftrightarrow$ 3. For all $i, j, k \in \{1, \dots, n\}$, we have

$$
\begin{aligned}
&E_k(g_{ij}) - \sum_{\ell=1}^n ({\Gamma_{ki}}^\ell\cdot g_{\ell j} + {\Gamma_{kj}}^\ell \cdot g_{i\ell}) \\
&= \nabla^U_{E_k}\langle E_i, E_j\rangle_g - \sum_{\ell=1}^n \big({\Gamma_{ki}}^\ell\cdot \langle E_\ell, E_j\rangle_g + {\Gamma_{kj}}^\ell \cdot \langle E_i, E_\ell\rangle_g\big) && (\text{Why?}) \\
&= \nabla^U_{E_k}\langle E_i, E_j\rangle_g - \big(\langle \nabla_{E_k} E_i, E_j\rangle_g + \langle E_i, \nabla_{E_k} E_j\rangle_g\big). && \text{(by definition of the connection coefficients)}
\end{aligned}
$$

This shows that 2. and 3. are equivalent. □

Secondly, we introduce the symmetry condition: The Lie bracket (Remark 2.3.10) is a *canonical* way of measuring the difference between the two different orders of differentiating two vector fields along each other. Connections allow to make sense of a single order of such a differentiation. The symmetry condition asks that the corresponding difference coincides with the Lie bracket:

Definition 2.3.9 (symmetric connection)**.** Let M be a smooth manifold. A linear connection ∇ on M is *symmetric* (or *torsion-free*) if the following holds: For all $X, Y \in \Gamma(\mathrm{T}\,M)$, we have

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

where $[X, Y]$ denotes the Lie bracket of X and Y (Remark 2.3.10).

Remark 2.3.10 (Lie bracket)**.** Let M be a smooth manifold and let $X, Y \in \Gamma(\mathrm{T}\,M)$. If $x \in M$, then

$$\begin{aligned} C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto X(x)\bigl(Y(x)(f)\bigr) - Y(x)\bigl(X(x)(f)\bigr) \end{aligned}$$

is a derivation at x (check!) and thus defines a tangent vector $[X, Y](x)$ at x. These tangent vectors combine into a vector field

$$[X, Y] \in \Gamma(\mathrm{T}\,M),$$

the *Lie bracket* of X and Y.

If $(E_i)_{i \in \{1,\dots,n\}}$ is a *coordinate* frame for $\mathrm{T}\,M$ on a chart domain U of M, then (check!)

$$\forall_{x \in U} \quad \forall_{i,j \in \{1,\dots,n\}} \quad [E_i, E_j](x) = 0.$$

Remark 2.3.11 (torsion tensor)**.** Let M be a smooth manifold and let ∇ be a linear connection on M. Then

$$\begin{aligned} \Gamma(\mathrm{T}\,M) \times \Gamma(\mathbf{T}\,M) &\longrightarrow \Gamma(\mathbf{T}\,M) \\ (X, Y) &\longmapsto \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned}$$

is $C^\infty(M)$-multilinear (check!). The resulting $(2,1)$-tensor field (via Remark 2.3.4) is the *torsion tensor of* ∇. The geometric effect of a non-vanishing torsion tensor can, e.g., be seen in Example 2.3.14.

In contrast, $(X, Y) \longmapsto \nabla_X Y - \nabla_Y X$ in general is *not* $C^\infty(M)$-multilinear (check!) and thus does *not* define a tensor field on M.

Proposition 2.3.12 (symmetric connections, in coordinates)**.** *Let M be a smooth manifold of dimension n and let ∇ be a linear connection on M. Then the following are equivalent:*

1. *The connection ∇ is symmetric.*

2. *For all coordinate frames $(E_i)_{i\in\{1,\dots,n\}}$ on a smooth chart domain U, we have*

$$\forall_{i,j\in\{1,\dots,n\}} \quad \nabla^U_{E_i} E_j - \nabla^U_{E_j} E_i = 0.$$

3. *For all coordinate frames of $\mathrm{T}\, M$ on a smooth chart domain, we have*

$$\forall_k \quad \forall_{i,j} \quad {\Gamma_{ij}}^k = {\Gamma_{ji}}^k,$$

where $({\Gamma_{ij}}^k)_{i,j,k\in\{1,\dots,n\}}$ denote the connection coefficients with respect to this coordinate frame.

Proof. All three properties are local (check!) and thus we only need to consider the case of a single smooth chart. Let $(E_i)_{i\in\{1,\dots,n\}}$ be a coordinate frame on a smooth chart domain U and let $({\Gamma_{ij}}^k)_{i,j,k}$ be the associated connection coefficients.

Ad 1. $\Longleftrightarrow$ 2. Because $(E_i)_{i\in\{1,\dots,n\}}$ is a *coordinate* frame, we obtain

$$\forall_{i,j\in\{1,\dots,n\}} \quad [E_i, E_j] = \boxed{?}$$

(Remark 2.3.10). Thus, if ∇ is symmetric, then the second condition is satisfied. Conversely, if the second condition is satisfied, then rewriting general vector fields in terms of the coordinate frame $(E_i)_{i,j\in\{1,\dots,n\}}$ and applying the various product rules shows that ∇ is symmetric (check!).

Ad 2. $\Longleftrightarrow$ 3. By definition of the connection coefficients, we have

$$\forall_{i,j\in\{1,\dots,n\}} \quad \sum_{k=1}^n ({\Gamma_{ij}}^k - {\Gamma_{ji}}^k) \cdot E_k = \nabla^U_{E_i} E_j - \nabla^U_{E_j} E_i.$$

Because the $(E_k)_{k\in\{1,\dots,n\}}$ form a basis at each point, we conclude that the vanishing of the sum on the left-hand side is equivalent to the vanishing of all summands. Thus, the second and the third property are equivalent. ☐

Example 2.3.13. If $n \in \mathbb{N}$, then the Euclidean linear connection on $\mathbb{R}^n$ is compatible with the Euclidean Riemannian metric (check!) and symmetric (check!).

Example 2.3.14 (a non-symmetric connection)**.** On $\mathbb{R}^3$, we consider the linear connection ∇ that is given by

$$\begin{array}{lll} \nabla_X X = 0 & \nabla_Y X = -Z & \nabla_Z X = Y \\ \nabla_X Y = Z & \nabla_Y Y = 0 & \nabla_Z Y = -X \\ \nabla_X Z = -Y & \nabla_Y X = X & \nabla_Z Z = 0 \end{array}$$

in terms of the standard coordinate frame (X, Y, Z) of $\mathrm{T}\,\mathbb{R}^3$. Then:

- The geodesics with respect to this connection ∇ coincide with the geodesics with respect to the Euclidean linear connection on $\mathbb{R}^3$ (Exercise).

- This connection is compatible with respect to the Euclidean Riemannian metric on $\mathbb{R}^3$ (check!). Moreover, the behaviour of the parallel transport gives an indication on how the torsion tensor of ∇ is related to a torsion/rotation phenomenon.
- This connection is *not* symmetric (check!).

Example 2.3.15. Let $N \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a smooth submanifold. We equip M with the Riemannian metric induced from the Euclidean Riemannian metric on $\mathbb{R}^N$ and with the linear connection $\nabla^\top$ induced by the Euclidean linear connection $\overline{\nabla}$ on $\mathbb{R}^N$ (Example 2.2.8). Then, $\nabla^\top$ is compatible with this Riemannian metric on M and symmetric, as the following computations show: Let $p\colon \mathrm{T}\,\mathbb{R}^N \longrightarrow \mathrm{T}\,M$ denote the orthogonal projection, let $X, Y, Z \in \Gamma(\mathrm{T}\,M)$, and let $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(\mathrm{T}\,\mathbb{R}^n)$ be extensions of X, Y, Z:

- *Compatibility:* Then, we have

$$
\begin{aligned}
\nabla^\top_X \langle Y, Z\rangle_2 &= X\bigl(\langle Y, Z\rangle_2\bigr) && \text{(by definition of the connection on functions)}\\
&= \bigl(\widetilde{X}\bigl(\langle \widetilde{Y}, \widetilde{Z}\rangle_2\bigr)\bigr)|_M && \text{(as } \widetilde{X}, \widetilde{Y}, \widetilde{X} \text{ extend } X, Y, Z\text{)}\\
&= \bigl(\overline{\nabla}_{\widetilde{X}} \langle \widetilde{Y}, \widetilde{Z}\rangle_2\bigr)|_M && \text{(by definition of the connection on functions)}\\
&= \langle \overline{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}\rangle_2|_M + \langle \widetilde{Y}, \overline{\nabla}_{\widetilde{X}} \widetilde{Z}\rangle_2|_M && \text{(compatibility of } \overline{\nabla}\text{; Example 2.3.13)}\\
&= \bigl\langle p(\overline{\nabla}_{\widetilde{X}} \widetilde{Y}), \widetilde{Z}\bigr\rangle_2|_M + \bigl\langle \widetilde{Y}, p(\overline{\nabla}_{\widetilde{X}} \widetilde{Z})\bigr\rangle_2|_M && \text{(on } M\text{, the fields } \widetilde{Y} \text{ and } \widetilde{Z} \text{ land in } \mathrm{T}\,M\text{)}\\
&= \bigl\langle p(\overline{\nabla}_{\widetilde{X}} \widetilde{Y}), Z\bigr\rangle_2 + \bigl\langle Y, p(\overline{\nabla}_{\widetilde{X}} \widetilde{Z})\bigr\rangle_2\\
&= \langle \nabla^\top_X Y, Z\rangle_2 + \langle Y, \nabla^\top_X Z\rangle_2.
\end{aligned}
$$

- *Symmetry:* We compute

$$
\begin{aligned}
\nabla^\top_X Y - \nabla^\top_Y X - [X, Y] &= p(\overline{\nabla}_{\widetilde{X}} \widetilde{Y} - \overline{\nabla}_{\widetilde{Y}} \widetilde{X}) - [X, Y]\\
&= p\bigl([\widetilde{X}, \widetilde{Y}]|_M\bigr) - [X, Y] && \text{(symmetry of } \overline{\nabla}\text{; Example 2.3.13)}\\
&= [X, Y] - [X, Y] && \text{(as } \widetilde{X}, \widetilde{Y} \text{ extend } X, Y\text{)}\\
&= 0.
\end{aligned}
$$

Quick check 2.3.16 (on the definition of symmetry)**.** In Definition 2.3.9:

1. What justifies the terminology "symmetric"?

   Hint

2. What justifies the terminology "torsion-free"?

   Hint

Theorem 2.3.17 (fundamental theorem of Riemannian geometry). *Every Riemannian manifold admits a unique linear connection that is compatible with the given Riemannian metric and that is symmetric. This connection is called* Levi-Civita connection*.*

Proof. Let (M, g) be a Riemannian manifold of dimension n. We proceed in the following steps (similar to the proof of the existence and uniqueness of covariant derivatives along curves; Theorem 2.2.16):

① *Koszul formula.* If ∇ is a linear connection on M that is compatible with g and symmetric, then

$$\begin{aligned}\langle \nabla_X Y, Z\rangle_g = \frac{1}{2} \cdot \bigl(&X(\langle Y, Z\rangle_g) + Y(\langle Z, X\rangle_g) - Z(\langle X, Y\rangle_g)\\ &- \langle Y, [X, Z]\rangle_g - \langle Z, [Y, X]\rangle_g + \langle X, [Z, Y]\rangle_g\bigr)\end{aligned}$$

holds (inside $C^\infty(M)$) for all $X, Y, Z \in \Gamma(\mathrm{T}\, M)$.

② We derive uniqueness from the Koszul formula.

③ From the Koszul formula, we also obtain the *local coordinate formula:* Let ∇ be a linear connection on M that is compatible with g and symmetric. Let $\varphi \colon U \longrightarrow U'$ be a smooth chart for M, let $({\Gamma_{ij}}^k)_{i,j,k \in \{1,\dots,n\}}$ be the connection coefficients with respect to the coordinate frame $(E_1, \dots, E_n)$ associated with φ, let $(g_{jk})_{j,k\in\{1,\dots,n\}}$ denote the coefficients of g with respect to φ, and let $(g^{jk})_{j,k\in\{1,\dots,n\}}$ be the coefficients of the pointwise inverse matrices. Then

$$\forall_{i,j,k \in \{1,\dots,n\}} \quad {\Gamma_{ij}}^k = \frac{1}{2} \cdot \sum_{\ell=1}^n g^{k\ell} \cdot \bigl(E_i(g_{j\ell}) + E_j(g_{i\ell}) - E_\ell(g_{ij})\bigr).$$

④ We use the local coordinate formula from ③ to define connections on chart domains U. The corresponding connections will be symmetric and compatible with $g|_U$.

⑤ We use uniqueness to see that the local constructions from ④ fit together. The resulting object is a linear connection on M that is compatible with g and symmetric.

Proof of ①. We prove this by direct computation: If ∇ is compatible with g and symmetric, then we obtain for all $X, Y, Z \in \Gamma(\mathrm{T}\, M)$ that

$$\begin{aligned} X\bigl(\langle Y, Z\rangle_g\bigr) &= \nabla_X \langle Y, Z\rangle_g \\ &= \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g && (\nabla \text{ is compatible with } g)\\ &= \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_Z X\rangle_g + \bigl\langle Y, [X, Z]\bigr\rangle_g, && (\text{symmetry of } \nabla)\end{aligned}$$

as well as analogous equations for "YZX" and "ZXY". Then

$$\text{equation for “}XYZ\text{”} + \text{equation for “}YZX\text{”} - \text{equation for “}ZXY\text{”}$$

and solving for $\langle \nabla_X Y, Z\rangle$ gives the Koszul formula (check!).

Proof of ②. The right-hand side of the Koszul formula is independent of the chosen connection. Thus, we can argue as follows: If ∇^1 and ∇^2 both are compatible with g and symmetric, then we obtain for all $X, Y \in \Gamma(\mathrm{T}\, M)$ that

$$\forall_{Z \in \Gamma(\mathrm{T}\, M)} \quad \langle \nabla^1_X Y - \nabla^2_X Y, Z\rangle_g = 0.$$

Taking $Z = \nabla^1_X Y - \nabla^2_X Y$ shows that $\nabla^1_X Y = \nabla^2_X Y$ (because). This completes the proof of uniqueness.

Proof of ③. Because $(E_1, \dots, E_n)$ is a coordinate frame, $[E_i, E_j] = 0$ for all $i, j \in \{1, \dots, n\}$ (Remark 2.3.10). Therefore, we obtain for all $i, j, \ell \in \{1, \dots, n\}$ from the Koszul formula that

$$\begin{aligned}
\sum_{k=1}^n {\Gamma_{ij}}^k \cdot g_{k\ell} &= \sum_{k=1}^n {\Gamma_{ij}}^k \cdot \langle E_k, E_\ell\rangle_g && \text{(definition of } g_{k\ell}\text{)}\\
&= \langle \nabla_{E_i} E_j, E_\ell\rangle_g && \text{(definition of } {\Gamma_{ij}}^k\text{)}\\
&= \frac{1}{2} \cdot \bigl(E_i(\langle E_j, E_\ell\rangle_g) + E_j(\langle E_\ell, E_i\rangle_g) - E_\ell(\langle E_i, E_j\rangle_g) - 0\bigr) && \text{(Koszul formula)}\\
&= \frac{1}{2} \cdot \bigl(E_i(g_{j\ell}) + E_j(g_{\ell i}) - E_\ell(g_{ij})\bigr). && \text{(definition of } g_{rs}\text{)}
\end{aligned}$$

In order to extract the connection coefficients, we now multiply with the inverse $(g^{\ell k})_{\ell,k}$ of $(g_{k\ell})_{k,\ell}$ from the right. For $i, j, k \in \{1, \dots, k\}$, this leads to

$$\begin{aligned}
{\Gamma_{ij}}^k &= \sum_{m=1}^n \sum_{\ell=1}^n {\Gamma_{ij}}^m \cdot g_{m\ell} \cdot g^{\ell k} && \text{(the matrices are mutually inverse)}\\
&= \sum_{\ell=1}^n g^{\ell k} \cdot \sum_{m=1}^n {\Gamma_{ij}}^m \cdot g_{m\ell} && \text{(the ring of smooth functions is commutative)}\\
&= \sum_{\ell=1}^n g^{\ell k} \cdot \frac{1}{2} \cdot \bigl(E_i(g_{j\ell}) + E_j(g_{\ell i}) - E_\ell(g_{ij})\bigr) && \text{(previous computation)}\\
&= \frac{1}{2} \cdot \sum_{\ell=1}^n g^{k\ell} \cdot \bigl(E_i(g_{j\ell}) + E_j(g_{i\ell}) - E_\ell(g_{ij})\bigr). && \text{(the matrices are symmetric)}
\end{aligned}$$

Proof of ④. We define a connection ∇^U on U via the local coordinate formula ③. This connection is

- symmetric: The right-hand side of the local coordinate formula is symmetric in “i” and “j”. Therefore, ∇^U has symmetric connection coefficients and is thus symmetric (Proposition 2.3.12).

- compatible with $g|_U$: The (proof of the) local coordinate formula ③ shows that the local characterisation of compatibility is satisfied (Proposition 2.3.8; check!).

Proof of ⑤. There is nothing more to say. This finishes the proof of existence. □

Definition 2.3.18 (Christoffel symbols)**.** The connection coefficients of Levi-Civita connections are also called *Christoffel symbols*.

Example 2.3.19 (Levi-Civita connections)**.**

- Let $n \in \mathbb{N}$. Then the Euclidean linear connection on $\mathbb{R}^n$ is the Hint

  .

  In particular, with respect to the standard coordinate frame the Christoffel symbols of $\mathbb{R}^n$ are all 0.

- Let $N \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a smooth submanifold. Then the submanifold connection (Example 2.2.8) is the Levi-Civita connection on M (Example 2.3.15).

- In particular: The previous example applies to the round spheres.

Proposition 2.3.20 (naturality of the Levi-Civita connection)**.** *Let* (M_1, g_1), (M_2, g_2) *be Riemannian manifolds with Levi-Civita connections* ∇^1 *and* ∇^2*, respectively, and let* $\varphi\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ *be an isometry, Then*

$$\varphi^*\nabla^2 = \nabla^1,$$

where the pullback $\varphi^*\nabla^2$ *is defined as in Proposition 2.2.4.*

Proof. In view of uniqueness of the Levi-Civita connection, we only need to show that the pullback $\nabla := \varphi^*\nabla^2$ is compatible with g_1 and symmetric. Both properties can be established through straightforward computations:

- *Compatibility:* For all $X, Y, Z \in \Gamma(\mathrm{T}\, M_1)$, we have

$$\begin{aligned}
\nabla_X \langle Y, Z\rangle_{g_1} &= X\bigl(\langle Y, Z\rangle_{g_1}\bigr) \\
&= X\bigl(\langle \varphi_* Y, \varphi_* Z\rangle_{g_2} \circ \varphi\bigr) && (\varphi \text{ is an isometry}) \\
&= (\varphi_* X)\bigl(\langle \varphi_* Y, \varphi_* Z\rangle_{g_2}\bigr) \circ \varphi && (\text{chain rule}) \\
&= \nabla^2_{\varphi_* X}\bigl(\langle \varphi_* Y, \varphi_* Z\rangle_{g_2}\bigr) \circ \varphi \\
&= \langle \nabla^2_{\varphi_* X} \varphi_* Y, \varphi_* Z\rangle_{g_2} \circ \varphi + \langle \varphi_* Y, \nabla^2_{\varphi_* X} \varphi_* Z\rangle_{g_2} \circ \varphi && (\nabla^2 \text{ is compatible with } g_2) \\
&= \langle (\varphi^{-1})_* \nabla^2_{\varphi_* X} \varphi_* Y, Z\rangle_{g_1} + \langle Y, (\varphi^{-1})_* \nabla^2_{\varphi_* X} \varphi_* Z\rangle_{g_1} && (\varphi^{-1} \text{ is an isometry}) \\
&= \langle \nabla_X Y, Z\rangle_{g_1} + \langle Y, \nabla_X Y\rangle_{g_1}. && (\text{definition of } \nabla = \varphi^*\nabla^2)
\end{aligned}$$

- *Symmetry:* For all $X, Y \in \Gamma(\mathrm{T}\, M_1)$, we have

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= (\varphi^{-1})_*(\nabla^2_{\varphi_* X}\varphi_* Y) - (\varphi^{-1})_*(\nabla^2_{\varphi_* Y}\varphi_* X) && \text{(definition of } \nabla = \varphi^*\nabla^2\text{)}\\
&= (\varphi^{-1})_*(\nabla^2_{\varphi_* X}\varphi_* Y - \nabla^2_{\varphi_* Y}\varphi_* X)\\
&= (\varphi^{-1})_*\bigl([\varphi_* X, \varphi_* Y]\bigr) && (\nabla^2 \text{ is symmetric})\\
&= [X, Y], && \text{(check!)}
\end{aligned}$$

which shows that ∇ is symmetric. □

Outlook 2.3.21 (Ehresmann connections)**.** Levi-Civita connections can also be viewed as extracting the appropriate part of the double tangent bundle:

Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection on M with respect to g. Moreover, let $\pi\colon \mathrm{T}\, M \longrightarrow M$ be the bundle projection map of the tangent bundle of M. Then the double tangent bundle $\mathrm{T}(\mathrm{T}\, M)$ contains a canonical "vertical" part, namely

$$\ker(\mathrm{d}\, \pi) \subset \mathrm{T}(\mathrm{T}\, M).$$

However, for general smooth manifolds, there is no *canonical* way to select a complementing "horizontal" part inside $\mathrm{T}(\mathrm{T}\, M)$. *Ehresmann connections* make such a selection; this concept can be formalised by smooth maps $c\colon \mathrm{T}(\mathrm{T}\, M) \longrightarrow \mathrm{T}\, M$ with appropriate properties that formalise this splitting/selection (that can also be enriched with the compatibility with g). In this language, one then has

$$\forall_{X,Y \in \Gamma(\mathrm{T}\, M)} \quad \nabla_X Y = c \circ \mathrm{d}\, Y \circ X.$$

2.4 Curvature tensors

We will measure the curvature of high-dimensional manifolds in terms of curvature of two-dimensional pieces. We will use the Levi-Civita connection (and its parallel transport) as basic input.

In terms of parallel transport, curvature can be introduced as follows: For linearly independent tangent vectors $v, w \in \mathrm{T}_x\, M$, one can find for $\varepsilon \longrightarrow 0$ smooth maps $G\colon [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}$ with $G(0,0) = x$ such that $G(\,\cdot\,, 0)$ represents v and $G(0,\,\cdot\,)$ represents w. Then, one measures the difference between the parallel transport along the "rectangle"

- first take $G(0,\,\cdot\,)$ from $x = G(0,0)$ to $G(0,\varepsilon)$,
- then take $G(\,\cdot\,,\varepsilon)$ from $G(0,\varepsilon)$ to $G(\varepsilon,\varepsilon)$,
- then take $G(\varepsilon, -\,\cdot\,)$ from $G(\varepsilon,\varepsilon)$ to $G(\varepsilon, 0)$,
- then take $G(-\,\cdot\,,\varepsilon)$ from $G(\varepsilon, 0)$ back to $G(0,0) = x$.

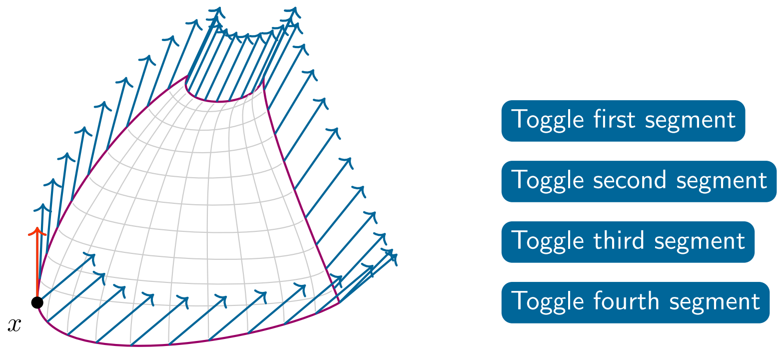


Figure 2.4.: Parallel transport along a rectangle, schematically

and the identity on $\mathrm{T}_x M$ (Figure 2.4) and takes $\varepsilon \longrightarrow 0$.

While this approach through parallel transport is geometrically satisfying, for most applications it is not the most convenient way of introducing curvature. Therefore, we will instead encode this idea as the difference between iterated covariant derivatives (with respect to the Levi-Civita connection) and the covariant derivative along the Lie bracket. This leads to the Riemannian curvature tensors.

These curvature tensors have four parameters and contain several redundancies. Therefore, one usually extracts various degrees of information from these tensors:

- sectional curvature
- Ricci curvature
- scalar curvature

As a first test case of whether the Riemannian curvature tensor models the correct notion, we show that Riemannian manifolds are locally isometric to Euclidean space if and only if the Riemannian curvature tensor is zero.

2.4.1 The Riemannian curvature tensor

Given vector fields X and Y, the Riemannian curvature tensor on a Riemannian manifold measures the difference between

- $\nabla_X \nabla_Y - \nabla_Y \nabla_X$ (the defect of switching the order of covariant differentiation) and
- $\nabla_{[X,Y]}$ (the baseline for such differences)

in terms of the Levi-Civita connection ∇.

Proposition and Definition 2.4.1 (the Riemannian curvature $(3,1)$-tensor). *Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. Then the map*

$$\begin{aligned}\mathrm{R}\colon \Gamma(\mathrm{T}\,M)\times\Gamma(\mathrm{T}\,M)\times\Gamma(\mathrm{T}\,M) &\longrightarrow \Gamma(\mathrm{T}\,M)\\ (X,Y,Z) &\longmapsto \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\end{aligned}$$

is $C^\infty(M)$-multilinear. The associated $(3,1)$-tensor field (Remark 2.3.4), which is also denoted by R*, is the* Riemannian curvature $(3,1)$-tensor of (M,g).

Proof. This is a computation, involving the various product rules. We prove $C^\infty(M)$-linearity in the third variable: For all $X,Y,Z\in\Gamma(\mathrm{T}\,M)$ and all $f\in C^\infty(M)$, we have

$$\begin{aligned}
\mathrm{R}(X,Y,f\cdot Z) &= \nabla_X\nabla_Y(f\cdot Z)-\nabla_Y\nabla_X(f\cdot Z)-\nabla_{[X,Y]}(f\cdot Z)\\
&= \nabla_X\big(Y(f)\cdot Z+f\cdot\nabla_Y Z\big)\\
&\quad-\nabla_Y\big(X(f)\cdot Z+f\cdot\nabla_X Z\big)\\
&\quad-[X,Y](f)\cdot Z-f\cdot\nabla_{[X,Y]}Z && \text{(product rule for }\nabla)\\
&= X(Y(f))\cdot Z+Y(f)\cdot Z+X(f)\cdot\nabla_Y Z+f\cdot\nabla_X\nabla_Y Z && \text{(product rule for }\nabla)\\
&\quad-Y(X(f))\cdot Z-X(f)\cdot\nabla_Y Z-Y(f)\cdot\nabla_X Z-f\cdot\nabla_Y\nabla_X Z && \text{(product rule for }\nabla)\\
&\quad-\big(X(Y(f))\cdot X-Y(X(f))\cdot Z\big)-f\cdot\nabla_{[X,Y]}Z && \text{(definition of }[X,Y])\\
&= f\cdot\big(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z\big) && \text{(cleanup)}\\
&= f\cdot\mathrm{R}(X,Y,Z).
\end{aligned}$$

The $C^\infty(M)$-linearity in the first two variables involves similar computations, using the product rule $[X,f\cdot Y]=f\cdot[X,Y]+X(f)\cdot Y$ for the Lie bracket. $\square$

Quick check 2.4.2 (on the definition of the Riemannian curvature tensor). In the defining formula for the $(3,1)$-Riemannian curvature tensor, g does not seem to appear. Where is it hidden?

Here!

Example 2.4.3 (Euclidean spaces have vanishing curvature). Let $n\in\mathbb{N}$. Then the Riemannian curvature $(3,1)$-tensor of $\mathbb{R}^n$ (with the Euclidean Riemannian metric) is 0:

The Levi-Civita connection on $\mathbb{R}^n$ is the Euclidean linear connection $\overline{\nabla}$ (Example 2.3.19).

Let $X,Y,Z\in\Gamma(\mathrm{T}\,\mathbb{R}^n)$, let $(E_1,\dots,E_n)$ be the standard coordinate frame on $\mathbb{R}^n$, and let $Z=\sum_{i=1}^n Z^i\cdot E_i$ be the representation of Z in this basis. Then, by definition of $\overline{\nabla}$ and the Lie bracket, we obtain

$$
\begin{aligned}
\overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z &= \overline{\nabla}_X \Big(\sum_{i=1}^n Y(Z^i) \cdot E_i \Big) - \overline{\nabla}_Y \Big(\sum_{i=1}^n X(Z^i) \cdot E_i \Big) \\
&= \sum_{i=1}^n \big(X(Y(Z^i)) - Y(X(Z^i)) \big) \cdot E_i \\
&= \sum_{i=1}^n [X,Y](Z^i) \cdot E_i \\
&= \overline{\nabla}_{[X,Y]} Z,
\end{aligned}
$$

and so $\mathrm{R}(X,Y,Z) = 0$. Alternatively, we can use the description in terms of Christoffel symbols (Remark 2.4.5).

Definition 2.4.4 (the Riemannian curvature $(4,0)$-tensor)**.** Let (M,g) be a Riemannian manifold. Then the map

$$
\begin{aligned}
\mathrm{Rm} \colon \Gamma(\mathrm{T}\,M)^{\times 4} &\longrightarrow C^\infty(M) \\
(X,Y,Z,W) &\longmapsto \big\langle \mathrm{R}(X,Y,Z), W \big\rangle_g
\end{aligned}
$$

is $C^\infty(M)$-multilinear (check!). The associated $(4,0)$-tensor field (Proposition 2.3.2), which is also denoted by Rm, is the *Riemannian* $(4,0)$*-tensor of* (M,g).

Remark 2.4.5 (the Riemannian curvature tensors, local description)**.** Let (M,g) be a Riemannian manifold of dimension n, let $\varphi \colon U \longrightarrow U'$ be a smooth chart for M, let $({\Gamma_{ij}}^k)_{i,j,k \in \{1,\dots,n\}}$ be the Christoffel symbols with respect to the coordinate frame $(E_1, \dots, E_n)$ associated with φ, and let $(g_{ij})_{i,j \in \{1,\dots,n\}}$ denote the coefficients of g with respect to φ. We obtain (check!):

- Let us write

  $$
  \mathrm{R} = \sum_{i,j,k,l \in \{1,\dots,n\}} {\mathrm{R}_{ijk}}^\ell \cdot \mathrm{d}\,x^i \otimes \mathrm{d}\,x^j \otimes \mathrm{d}\,x^k \otimes E_\ell,
  $$

  with ${\mathrm{R}_{ijk}}^\ell \in C^\infty(M)$, i.e., $\mathrm{R}(E_i, E_j, E_k) = \sum_{\ell=1}^n {\mathrm{R}_{ijk}}^\ell \cdot E_\ell$ (Remark 1.2.28). Then, for all $i,j,k \in \{1,\dots,n\}$, we have

  $$
  {\mathrm{R}_{ijk}}^\ell = E_i({\Gamma_{jk}}^\ell) - E_j({\Gamma_{ik}}^\ell) + \sum_{m=1}^n ({\Gamma_{jk}}^m \cdot {\Gamma_{im}}^\ell - {\Gamma_{ik}}^m \cdot {\Gamma_{jm}}^\ell).
  $$

  Using the local coordinate formula for the Christoffel symbols (proof of Theorem 2.3.17), this can be expressed in terms of the g_{ij} and the E_i.

- Furthermore, if we write

  $$
  \mathrm{Rm} = \sum_{i,j,k,l \in \{1,\dots,n\}} \mathrm{R}_{ijk\ell} \cdot \mathrm{d}\,x^i \otimes \mathrm{d}\,x^j \otimes \mathrm{d}\,x^k \otimes \mathrm{d}\,x^\ell
  $$

with $\mathrm{R}_{ijk\ell} \in C^\infty(M)$, then

$$\begin{aligned}\mathrm{R}_{ijk\ell} &= \bigl\langle \mathrm{R}(E_i, E_j, E_k), E_\ell \bigr\rangle_g = \sum_{m=1}^n \langle \mathrm{R}_{ijk}{}^m \cdot E_m, E_\ell \rangle_g \\ &= \sum_{m=1}^n \mathrm{R}_{ijk}{}^m \cdot g_{m\ell} = \sum_{m=1}^n g_{\ell m} \cdot \mathrm{R}_{ijk}{}^m \\ &= \sum_{m=1}^n g_{\ell m} \cdot \Bigl(E_i(\Gamma_{jk}{}^m) - E_j(\Gamma_{ik}{}^m) + \sum_{p=1}^n (\Gamma_{jk}{}^p \cdot \Gamma_{ip}{}^m - \Gamma_{ik}{}^p \cdot \Gamma_{jp}{}^m) \Bigr).\end{aligned}$$

Achievement unlocked!

Remark 2.4.6 (raising and lowering indices)**.** The construction in Definition 2.4.4 is an instance of *lowering indices* (see the local notation in Remark 2.4.5!): Let (M, g) be a Riemannian manifold and let $k, \ell \in \mathbb{N}$. If $F\colon \Gamma(\mathrm{T}\, M)^{\times k} \times \Gamma(\mathrm{T}^*\, M)^{\times \ell} \longrightarrow \Gamma(\mathrm{T}\, M)$ is $C^\infty(M)$-multilinear, then

$$\begin{aligned} F^\flat \colon \Gamma(\mathrm{T}\, M)^{\times k} \times \Gamma(\mathrm{T}\, M)^{\times \ell+1} &\longrightarrow C^\infty(M) \\ (X_1, \dots, X_k, Y_1, \dots, Y_{\ell+1}) &\longmapsto \bigl\langle F(X_1, \dots, X_k, Y_1, \dots, Y_\ell), Y_{\ell+1} \bigr\rangle_g \end{aligned}$$

is $C^\infty(M)$-multilinear (check!). In this notation, we have

$$\mathrm{Rm} = \mathrm{R}^\flat\,.$$

Conversely, because g is non-degenerate, one can also *raise indices*: If $F\colon \Gamma(\mathrm{T}\, M)^{\times k} \times \Gamma(\mathrm{T}^*\, M)^{\times \ell+1} \longrightarrow C^\infty(M)$ is $C^\infty(M)$-multilinear, then there exists a unique $C^\infty(M)$-multilinear map $F^\sharp$ with $F = (F^\sharp)^\flat$ (check!). In this notation, we have

$$\mathrm{R} = \mathrm{Rm}^\sharp\,.$$

The conversions $\flat$ ("flat") and $\sharp$ ("sharp") are also called *musical isomorphisms*. As can be seen in Remark 2.4.5, local descriptions of the raised/lowered versions differ by the matrix coefficients of the Riemannian metric!

A key property of the Riemannian curvature tensor is that it is an invariant of Riemannian geometry:

Proposition 2.4.7 (naturality of the Riemannian curvature tensor)**.** *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds with Riemannian curvature $(3,1)$-tensors R_1 and R_2, respectively, and let $\varphi\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be an isometry. Then, for all $X, Y, Z \in \Gamma(\mathrm{T}\, M_1)$, we have*

$$\varphi_*\bigl(\mathrm{R}_1(X, Y, Z)\bigr) = \mathrm{R}_2(\varphi_* X, \varphi_* Y, \varphi_* Z).$$

Moreover, for all $X, Y, Z, W \in \Gamma(\mathrm{T}\, M)$*, we have*

$$\mathrm{Rm}_1(X, Y, Z, W) = \mathrm{Rm}_2(\varphi_* X, \varphi_* Y, \varphi_* Z, \varphi_* W) \circ \varphi.$$

Proof. This follows from a straightforward computation, using the naturality of the Levi-Civita connection (Proposition 2.3.20) and the definition of the Riemannian curvature tensor (check!). □

Remark 2.4.8 (local isometries and the Riemannian curvature tensor)**.** There is also a corresponding version of Proposition 2.4.7 for local isometries: All relevant terms are local (check!).

For computations with the Riemannian curvature tensor, we spell out some of the built-in redundancies:

Proposition 2.4.9 (symmetries of the Riemannian curvature tensor)**.** *Let* (M, g) *be a Riemannian manifold. Then, for all* $X, Y, Z, W \in \Gamma(\mathrm{T}\, M)$*, we have*

1. $\mathrm{Rm}(W, X, Y, Z) = -\mathrm{Rm}(X, W, Y, Z)$
2. $\mathrm{Rm}(W, X, Y, Z) = -\mathrm{Rm}(W, X, Z, Y)$
3. $\mathrm{Rm}(W, X, Y, Z) = \mathrm{Rm}(Y, Z, W, X)$
4. (First) Bianchi identity.

$$\mathrm{Rm}(W, X, Y, Z) + \mathrm{Rm}(X, Y, W, Z) + \mathrm{Rm}(Y, W, X, Z) = 0.$$

Proof. Let ∇ denote the Levi-Civita connection on (M, g).

Ad 1. This is a direct consequence of the definition (check!).

Ad 2. This can be derived from the compatibility of ∇ with g: By the usual argument from multilinear algebra, it suffices to show that $\mathrm{Rm}(W, X, Y, Y) = 0$ (check!). By definition,

$$\begin{aligned}\mathrm{Rm}(W, X, Y, Y) &= \big\langle \mathrm{R}(W, X, Y), Y \big\rangle_g \\ &= \langle \nabla_W \nabla_X Y, Y \rangle_g - \langle \nabla_X \nabla_W Y, Y \rangle_g - \langle \nabla_{[W,X]} Y, Y \rangle_g.\end{aligned}$$

Using the compatibility of ∇ with g, we can rewrite these terms as follows:

- We have

$$\begin{aligned}\langle \nabla_W \nabla_X Y, Y \rangle_g + \langle \nabla_X Y, \nabla_W Y \rangle_g &= \nabla_W \langle \nabla_X Y, Y \rangle_g && \text{(compatibility)} \\ &= \nabla_W \Big(\frac{1}{2} \cdot \nabla_X \langle Y, Y \rangle_g \Big) && \text{(compatibility; symmetry of } g\text{)} \\ &= \frac{1}{2} \cdot W\big(X(\|Y\|_g^2)\big). && \text{(definition of } \nabla_W \text{ on } C^\infty(M)\text{)}\end{aligned}$$

- Analogously, $\langle \nabla_X \nabla_W Y, Y \rangle_g + \langle \nabla_W Y, \nabla_X Y \rangle = \frac{1}{2} \cdot X\big(W(\|Y\|_g^2)\big)$.

- Finally,

$$\begin{aligned}
[W,X](\|Y\|_g^2) &= \nabla_{[W,X]}\langle Y,Y\rangle_g && \text{(definition of } \nabla_{[W,X]} \text{ on } C^\infty(M))\\
&= \langle \nabla_{[W,X]}Y, Y\rangle_g + \langle Y, \nabla_{[W,X]}Y\rangle_g && \text{(compatibility)}\\
&= 2\cdot \langle \nabla_{[W,X]}Y, Y\rangle_g.
\end{aligned}$$

Plugging in these expressions results in

$$\begin{aligned}
\operatorname{Rm}(W,X,Y,Y) &= \frac{1}{2}\cdot\bigl(W(X(\|Y\|_g^2)) - X(W(\|Y\|_g^2)) - [W,X](\|Y\|_g^2)\bigr)\\
&\quad - \frac{1}{2}\cdot\langle \nabla_X Y, \nabla_W Y\rangle_g + \frac{1}{2}\cdot\langle \nabla_W Y, \nabla_X Y\rangle_g\\
&= 0 + 0. && \text{(definition of } [W,X]\text{; symmetry of } g)
\end{aligned}$$

Ad 4. The key players in this computation are the Jacobi identitiy for the Lie bracket and the symmetry of ∇: Expanding the definition of Rm and using the symmetry of the connection, we obtain (check!)

$$\begin{aligned}
\mathrm{R}(W,X,Y) + \mathrm{R}(X,Y,W) + \mathrm{R}(Y,W,X) &= \bigl[W,[X,Y]\bigr] + \bigl[X,[Y,W]\bigr] + \bigl[Y,[W,X]\bigr]\\
&= 0 && \text{(Jacobi identity)}
\end{aligned}$$

From this identity for R, we directly obtain the Bianchi identity for Rm.

Ad 3. We derive this from the other three identities: The Bianchi identity (applied four times) gives

$$\begin{aligned}
\operatorname{Rm}(W,X,Y,Z) + \operatorname{Rm}(X,Y,W,Z) + \operatorname{Rm}(Y,W,X,Z) &= 0\\
\operatorname{Rm}(X,Y,Z,W) + \operatorname{Rm}(Y,Z,X,W) + \operatorname{Rm}(Z,X,Y,W) &= 0\\
\operatorname{Rm}(Y,Z,W,X) + \operatorname{Rm}(Z,W,Y,X) + \operatorname{Rm}(W,Y,Z,X) &= 0\\
\operatorname{Rm}(Z,W,X,Y) + \operatorname{Rm}(W,X,Z,Y) + \operatorname{Rm}(X,Z,W,Y) &= 0.
\end{aligned}$$

Therefore, also the sum of these four equations adds up to zero. The first two columns add up to zero: We can apply the second part four times. Applying the first two parts shows that the last column adds up to

$$2\cdot \operatorname{Rm}(Y,W,X,Z) - 2\cdot \operatorname{Rm}(X,Z,Y,W).$$

Hence, this term is also zero, which proves the claim. □

Quick check 2.4.10 (symmetries of the Riemannian curvature tensor, locally)**.** In the situation of Proposition 2.4.9, how can the symmetries of the Riemannian curvature tensor be expressed locally in terms of coefficients $\mathrm{R}_{ijk\ell}$?

1. Solution

2. Solution

3. Solution

4. Solution

2.4.2 Flat manifolds

We now verify that the vanishing of the Riemannian curvature tensor has the correct geometric meaning; the Riemannian curvature tensor is zero if and only if the Riemannian manifold is locally isometric to Euclidean space.

Definition 2.4.11 (flat manifold)**.** A Riemannian manifold is *flat* if its Riemannian curvature tensor is zero.

Quick check 2.4.12. In Definition 2.4.11, does it matter whether we use the Riemannian curvature $(3,1)$-tensor or the Riemannian curvature $(4,0)$-tensor?
Yes No

Example 2.4.13 (flatness of Euclidean space)**.** Let $n \in \mathbb{N}$. Then, $\mathbb{R}^n$ with the Euclidean Riemannian metric is flat (Example 2.4.3).

Theorem 2.4.14 (flatness)**.** *A Riemannian manifold is flat if and only if it is locally isometric to a Euclidean space (with the Euclidean Riemannian metric).*

Proof. Euclidean spaces are flat (Example 2.4.13). Therefore, also all Riemannian manifolds that are locally isometric to Euclidean space (with the Euclidean Riemannian metric) are flat (Remark 2.4.8).

Conversely, let (M, g) be a flat Riemannian manifold of dimension n and let $x \in M$. Showing that (M, g) around x is isometric with an open subset of $\mathbb{R}^n$ with the Euclidean Riemannian metric amounts to finding suitable coordinates around x. For this, we need some preparations.

Definition 2.4.15 (parallel vector field)**.** A vector field on a Riemannian manifold is *parallel* if it is parallel (with respect to the Levi-Civita connection) along *all* smooth curves.

Because (M, g) is flat, we can find on a neighbourhood of x a parallel orthonormal frame: Let $(v_1, \dots, v_n)$ be an orthonormal basis of $\mathrm{T}_x M$. By Lemma 2.4.16, we can extend these vectors on some neighbourhood U of x to parallel vector fields $E_1, \dots, E_n$ with

$$\forall_{j \in \{1,\dots,n\}} \quad E_j(x) = v_j.$$

Because parallel transport with respect to the Levi-Civita connection is a linear isometry (Proposition 2.3.7), $(E_1, \dots, E_n)$ is an orthonormal frame on U.

In order to complete the proof, we now only need to show that this frame is a *coordinate* frame on some open neighbourhood of x (these coordinates provide the desired local isometry; check!).

In particular, for all $j, k \in \{1, \dots, n\}$, we obtain

$$\begin{aligned} [E_j, E_k] &= \nabla_{E_i} E_j - \nabla_{E_j} E_i, && (\text{Why?}) \\ &= 0 && (\text{Why?}) \end{aligned}$$

where ∇ denotes the Levi-Civita connection on (M, g). This means that the $(E_j)_{j \in \{1,\dots,n\}}$ are so-called commuting vector fields. Differential topology shows that there exists a neighbourhood $W \subset U$ of x on which this commuting frame is a coordinate frame (Exercise; this is a converse of Remark 2.3.10). This finishes the proof. $\square$

Lemma 2.4.16 (existence of parallel vector fields on flat manifolds)**.** *Let (M, g) be a flat Riemannian manifold, let $x \in M$, and let $v \in \mathrm{T}_x M$. Then, there exists a parallel vector field V on a neighbourhood of x with $V_x = v$.*

Proof. Let $n := \dim M$. By choosing local coordinates of M around x, for notational convenience, we may assume without loss of generality that (check!)

$$x = 0 \in (-1, 1)^n = M;$$

at this point, we don't know anything about the Riemannian metric (and the Levi-Civita connection) on this cube, except that its Riemannian curvature tensor is zero.

We construct our vector field inductively (Figure 2.5):

- We first apply parallel transport to v (at 0) along the first axis.
- From each point on the first axis, we apply parallel transport to the constructed vector along the second axis.
- From each point in the $(1, 2)$-plane, we apply parallel transport to the constructed vector along the third axis.
- ...

This leads to a vector field V on M with $V_x = v$; smoothness of V is a consequence of the fact that solutions to smooth linear ordinary differential equations smoothly depend on their initial conditions (check!).

By construction, V satisfies a partial parallelism condition. However, we need that V is everywhere parallel in all directions. In view of linearity of the connection in the first argument, it suffices to show that

$$\forall_{j \in \{1,\dots,n\}} \quad \nabla_{E_j} V = 0,$$

where $(E_j)_{j \in \{1,\dots,n\}}$ is the frame associated with the standard coordinates.

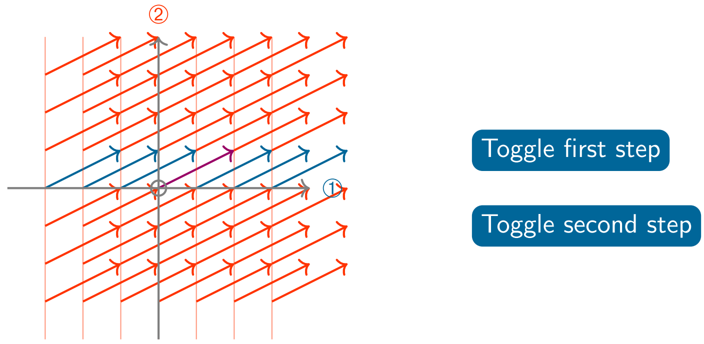


Figure 2.5.: Inductive construction of a parallel vector field, schematically

We prove this by induction: More precisely, for each $k \in \{1, \dots, n\}$ we show that on the subcube $M_k := (-1,1)^k \times \{0\}^{n-k} \subset M$, we have:

$$\forall_{j \in \{1,\dots,k\}} \quad \nabla_{E_j} V = 0.$$

The case $k = 1$ holds by construction of V. For the induction hypothesis, let us now suppose that the claim is proved for some $k \in \{1, \dots, n-1\}$. In particular, we have $\nabla_{E_j} V = 0$ on M_k for all $j \in \{1, \dots, k\}$ and (by construction) $\nabla_{E_{k+1}} V = 0$ on M_{k+1}. We need to extend the first k equations to all of M_{k+1}.

For all $j \in \{1, \dots, k\}$, we have $[E_{k+1}, E_j] = 0$ (Remark 2.3.10) and so on M_{k+1}, we have

$$\begin{aligned}
\nabla_{E_{k+1}} \nabla_{E_j} V &= \mathrm{R}(E_{k+1}, E_j, V) + \nabla_{E_j} \nabla_{E_{k+1}} V + \nabla_{[E_{k+1}, E_j]} V && \text{(definition of R)} \\
&= 0 + \nabla_{E_j} \nabla_{E_{k+1}} V + 0 && \text{(flatness; } [E_{k+1}, E_j] = 0\text{)} \\
&= 0 && \text{(as } \nabla_{E_{k+1}} V = 0 \text{ on } M_{k+1}\text{).}
\end{aligned}$$

This means that $\nabla_{E_j} V$ on M_{k+1} is parallel along the curves starting on M_k in the direction of the $(k+1)$-axis. Because $\nabla_{E_j} V$ is zero on M_k and because the zero vector field has the same parallelism property, the uniqueness of parallel vector fields along curves (with given initial vector) shows that $\nabla_{E_j} V = 0$ on all of M_{k+1}.

Hence, we obtain that $\nabla_{E_j} V = 0$ on $M_n = M$ for all $j \in \{1, \dots, n\}$. Thus, V is parallel. By construction, $V_x = v$. □

Example 2.4.17 (flat torus)**.** Let $n \in \mathbb{N}$. The torus $\mathbb{Z}^n \setminus \mathbb{R}^n$ (with the Riemannian metric induced from the Euclidean Riemannian metric on $\mathbb{R}^n$; Example 1.4.24) is locally isometric to the flat $\mathbb{R}^n$ (Proposition 1.4.20) and thus itself flat (Remark 2.4.8). But the 2-torus with the Riemannian metric inherited from the "standard" embedding into $\mathbb{R}^3$ is *not* flat (Exercise).

2.4.3 Sectional curvature

In order to reduce the inherent complexity and redundancies of the Riemannian curvature tensors, one reorganises the information contained in the Riemannian curvature tensor. This leads to sectional, Ricci, and scalar curvature.

The first such reduction is sectional curvature. It has a slightly more geometric flavour than the Riemannian curvature tensor, but it is a different type of object: It is not a tensor field. In short, sectional curvature assigns a real number to each plane in the tangent spaces (represented by two spanning vectors); geometrically, this is the Gauß curvature of the local surface determined by this tangent plane [18, Chapter 8].

For the definition of sectional curvature, we start with $\operatorname{Rm}(v, w, w, v)$ (which roughly measures the v-component of the parallel transport of w around the infinitesimal quadrangle spanned by v and w). In order to make this term independent of the spanning vectors v and w of the tangent plane, we need to divide out the following correction term:

Remark 2.4.18. Let V be a real vector space with inner product $\langle \cdot\,, \cdot \rangle$. For $v, w \in V$, we set

$$|v \wedge w| := \sqrt{\|v\|^2 \cdot \|w\|^2 - \langle v, w\rangle^2} \in \mathbb{R}_{\geq 0}.$$

It should be noted that the term under the square root is always non-negative and that it is non-zero if and only if v and w are linearly independent because

.

Geometrically, $|v \wedge w|$ is Hint

. In particular, if $v \perp w$ and $\|v\| = 1 = \|w\|$, then $|v \wedge w| = 1$. Similarly, if v and w are linearly independent and $\widetilde{v}, \widetilde{w} \in V$ span the same plane as v, w, then

$$|\widetilde{v} \wedge \widetilde{w}| = |\det A| \cdot |v \wedge w|,$$

where $A \in \operatorname{GL}(2, \mathbb{R})$ is the matrix describing $\widetilde{v}$ and $\widetilde{w}$ in terms of v and w (check!).

Definition 2.4.19 (sectional curvature)**.** Let (M, g) be a Riemannian manifold and let $x \in M$. Then the *sectional curvature of* (M, g) *at* x is given by: For all linearly independent vectors $v, w \in T_x M$, we set

$$\sec(v, w) := \frac{\operatorname{Rm}(v, w, w, v)}{|v \wedge w|_g^2} \in \mathbb{R}.$$

(This implicitly uses the conversion between multilinear maps and tensor fields!) If we want to emphasise the point x or the used Riemannian metric, then we write $\sec_x$, $\sec^g$, ... instead of sec.

Alternatively, one can also view sectional curvature as a section of the 2-Grassmannian bundle.

Example 2.4.20 (flat manifolds)**.** If (M, g) is a flat Riemannian manifold, then $\sec = 0$, i.e., for all $x \in M$ and all linearly independent $v, w \in \mathrm{T}_x M$, we have

$$\sec(v, w) = 0.$$

This follows from the fact that $\mathrm{Rm} = 0$, as (M, g) is flat.

In particular, Euclidean space (with the Euclidean Riemannian metric) and tori (with the Riemannian metric from Example 1.4.24) have constant sectional curvature 0.

Remark 2.4.21 (local isometries and sectional curvature)**.** Let $\varphi\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be a local isometry between Riemannian manifolds. Then, for all $x \in M_1$ and all linearly independent $v, w \in \mathrm{T}_x M_1$, we have

$$\sec_x^{M_1, g_1}(v, w) = \sec_{\varphi(x)}^{M_2, g_2}\bigl(\mathrm{d}_x \varphi(v), \mathrm{d}_x \varphi(w)\bigr),$$

because local isometries preserve both the numerator and the denominator in the definition of sectional curvature (Remark 2.4.8, Remark 2.4.18).

Proposition 2.4.22 (sectional curvature only depends on the tangent plane)**.** *Let (M, g) be a Riemannian manifold and let $x \in M$. Let $v, w \in \mathrm{T}_x M$ and $\widetilde{v}, \widetilde{w} \in \mathrm{T}_x M$ with the following properties:*

- *The pairs (v, w) and $(\widetilde{v}, \widetilde{w})$ each are linearly independent.*
- *The pairs (v, w) and $(\widetilde{v}, \widetilde{w})$ span the same plane in $\mathrm{T}_x M$.*

Then $\sec(v, w) = \sec(\widetilde{v}, \widetilde{w})$.

Proof. Because the pairs span the same plane, there exist $\alpha_v, \beta_v, \alpha_w, \beta_w \in \mathbb{R}$ with

$$\widetilde{v} = \alpha_v \cdot v + \beta_v \cdot w \quad \text{and} \quad \widetilde{w} = \alpha_w \cdot v + \beta_w \cdot w.$$

We now consider the denominator and the numerator of sectional curvature separately: We have $A := \begin{pmatrix} \alpha_v & \alpha_w \\ \beta_v & \beta_w \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ and (Remark 2.4.18)

$$|\widetilde{v} \wedge \widetilde{w}|_g = |\det A| \cdot |v \wedge w|_g.$$

On the other hand, the multilinearity of Rm and the symmetries of Rm show that

$$\begin{aligned}\mathrm{Rm}(\widetilde{v},\widetilde{w},\widetilde{w},\widetilde{v}) &= \alpha_v\beta_w\cdot\alpha_w\beta_v\cdot\mathrm{Rm}(v,w,v,w)+\alpha_v\beta_w\cdot\beta_w\alpha_v\cdot\mathrm{Rm}(v,w,w,v)\\ &\quad+\beta_v\alpha_w\cdot\alpha_w\beta_v\cdot\mathrm{Rm}(w,v,v,w)+\beta_v\alpha_w\cdot\beta_w\alpha_v\cdot\mathrm{Rm}(w,v,w,v)\\ &\quad+\text{terms containing }\mathrm{Rm}(u,u,\dots)\text{ or }\mathrm{Rm}(\dots,u,u)\\ &= (-\alpha_v\beta_w\cdot\alpha_w\beta_v+\alpha_v\beta_w\cdot\beta_w\alpha_v+\beta_v\alpha_w\cdot\alpha_w\beta_v-\beta_v\alpha_w\cdot\beta_w\alpha_v)\cdot\mathrm{Rm}(v,w,w,v)\\ &\quad+0 \qquad\qquad (\text{symmetries of Rm; Proposition 2.4.9})\\ &= \det A\cdot\det A\cdot\mathrm{Rm}(v,w,w,v).\end{aligned}$$

Thus, in total, we obtain

$$\sec(\widetilde{v},\widetilde{w}) = \frac{\mathrm{Rm}(\widetilde{v},\widetilde{w},\widetilde{w},\widetilde{v})}{|\widetilde{v}\wedge\widetilde{w}|_g^2} = \frac{(\det A)^2\cdot\mathrm{Rm}(v,w,w,v)}{|\det A|^2\cdot|v\wedge w|_g^2} = \sec(v,w). \qquad \square$$

Proposition 2.4.23 (sectional curvature determines Riemannian curvature). *Let M be a smooth manifold and let R_1, R_2 be $(4,0)$-tensor fields that satisfy the symmetries in Proposition 2.4.9. Moreover, for all $x \in M$ and all linearly independent $v, w \in \mathrm{T}_x M$ we assume that*

$$R_1(v,w,w,v) = R_2(v,w,w,v).$$

Then $R_1 = R_2$.

Proof. The difference $D := R_1 - R_2$ is a $(4,0)$-tensor field on M (check!) that satisfies

$$D(v,w,w,v) = 0$$

for all $x \in M$ and all $v, w \in \mathrm{T}_x M$ and that satisfies the symmetries in Proposition 2.4.9 (check!). Then, manipulations using these symmetries show that $D = 0$ (Exercise). $\square$

Corollary 2.4.24 (flatness via sectional curvature). *A Riemannian manifold has constant sectional curvature 0 if and only if it is flat.*

Proof. Flat manifolds clearly have constant sectional curvature 0 (Example 2.4.20). Conversely, if (M,g) is a Riemannian manifold with $\sec = 0$, then for all $x \in M$ and all linearly independent $v, w \in \mathrm{T}_x M$, we have

$$\mathrm{Rm}(v,w,w,v) = 0.$$

We can now apply Proposition 2.4.23 (check!) and obtain that $\mathrm{Rm} = 0$. Thus, (M,g) is flat. $\square$

2.4.4 Ricci curvature

Successively averaging over the available directions in the tangent spaces further condenses the curvature information down to Ricci curvature (which basically depends only on a single tangent vector) and scalar curvature (which depends only on the point). The advantage of these constructions is that the resulting curvature invariants are not as bulky as the Riemannian curvature tensor or sectional curvature; the disadvantage is that they do not contain as much information.

We start with Ricci curvature. We first introduce the Ricci tensor via traces and then show how Ricci curvature of a tangent vector can be interpreted as averaging over all sectional curvatures of the tangent planes containing the given tangent vector.

Definition 2.4.25 (Ricci curvature)**.** Let (M, g) be a Riemannian manifold. Then the *Ricci tensor of* (M, g) (or *Ricci curvature tensor*) is the $(2, 0)$-tensor defined from the $C^\infty(M)$-bilinear map

$$\begin{aligned}\mathrm{Ric}\colon \Gamma(\mathrm{T}\,M) \times \Gamma(\mathrm{T}\,M) &\longrightarrow C^\infty(M)\\ (X, Y) &\longmapsto \mathrm{tr}\bigl(Z \mapsto \mathrm{R}(Z, X, Y)\bigr).\end{aligned}$$

Example 2.4.26 (flat manifolds)**.** Flat manifolds have Ricci curvature zero.

Remark 2.4.27 (Ricci curvature)**.** Let (M, g) be a Riemannian manifold. Then Ric can be reconstructed from the map

$$\begin{aligned}\mathrm{T}\,M &\longrightarrow \mathbb{R}\\ \mathrm{T}_x\,M \ni v &\longmapsto \mathrm{Ric}(v, v)\end{aligned}$$

via polarisation (Exercise). This is the Ricci analogue of Proposition 2.4.23.

Proposition 2.4.28 (Ricci curvature via averaging)**.** *Let (M, g) be a Riemannian manifold of dimension n, let $x \in M$, and let $v \in \mathrm{T}_x\,M$. If $(v_1, \dots, v_n)$ is an orthonormal basis of $\mathrm{T}_x\,M$ with $v_1 = v$, then*

$$\mathrm{Ric}(v, v) = \sum_{k=2}^{n} \sec(v_k, v_1).$$

Proof. By construction,

$$\begin{aligned}\mathrm{Ric}(v, v) &= \mathrm{tr}\bigl(z \mapsto \mathrm{R}(z, v, v)\bigr)\\ &= \sum_{k=1}^{n} \bigl\langle \mathrm{R}(v_k, v, v), v_k \bigr\rangle_g && (\text{because} \qquad)\\ &= \sum_{k=1}^{n} \mathrm{Rm}(v_k, v, v, v_k) && (\text{definition of Rm})\\ &= \sum_{k=2}^{n} \mathrm{Rm}(v_k, v, v, v_k) && (\text{because } v_1 = v)\\ &= \sum_{k=2}^{n} \sec(v_k, v). && (\text{definition of sec and } |v_k \wedge v|_g = 1)\end{aligned}$$

Thus, Ric is the "average" of sec. The fact that on the right-hand side we only have $n-1$ summands leads to the appearance of $n-1$ in many formulas involving Ric. $\square$

Remark 2.4.29 (local isometries and Ricci curvature)**.** Let $\varphi\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be a local isometry between Riemannian manifolds. Then, for all $x \in M_1$ and all $v, w \in T_x M_1$, we have

$$\mathrm{Ric}_x^{M_1,g_1}(v,w) = \mathrm{Ric}_{\varphi(x)}^{M_2,g_2}\bigl(\mathrm{d}_x\,\varphi(v), \mathrm{d}_x\,\varphi(w)\bigr),$$

as local isometries preserve the Riemannian curvature tensor (Remark 2.4.8).

Study note. You now understand all terms of the Ricci flow equation on p. 3!

2.4.5 Scalar curvature

Scalar curvature is the trace of Ricci curvature. As the name suggests, scalar curvature is scalar-valued.

Definition 2.4.30 (scalar curvature)**.** Let (M, g) be a Riemannian manifold. Then the *scalar curvature of* (M, g) is the smooth map

$$\begin{aligned} \mathrm{scal}\colon M &\longrightarrow \mathbb{R} \\ x &\longmapsto \mathrm{tr}\bigl(Z \mapsto \mathrm{Ric}_x^\sharp(Z)\bigr). \end{aligned}$$

Here, $\mathrm{Ric}^\sharp$ is obtained from Ric by raising indices via g (Remark 2.4.6).

Example 2.4.31 (flat manifolds)**.** Flat manifolds have scalar curvature zero.

Proposition 2.4.32 (scalar curvature via averaging)**.** *Let (M, g) be a Riemannian manifold of dimension n, let $x \in M$, and let $(v_1, \dots, v_n)$ be an orthonormal basis of $T_x M$. Then*

$$\mathrm{scal}(x) = \sum_{j,k \in \{1,\dots,n\}, j \neq k} \sec(v_j, v_k).$$

Proof. By definition,

$$\begin{aligned} \mathrm{scal}(x) &= \mathrm{tr}\bigl(z \mapsto \mathrm{Ric}^\sharp(z)\bigr) \\ &= \sum_{j=1}^n \bigl\langle \mathrm{Ric}^\sharp(v_j), v_j \bigr\rangle_g && \text{(because } (v_1,\dots,v_n) \text{ is an ONB)} \\ &= \sum_{j=1}^n \mathrm{Ric}(v_j, v_j) && \text{(definition of } \mathrm{Ric}^\sharp) \\ &= \sum_{j=1}^n \sum_{k \in \{1,\dots,n\} \setminus \{j\}} \sec(v_j, v_k), && \text{(by Proposition 2.4.28)} \end{aligned}$$

as claimed. $\square$

Remark 2.4.33 (local isometries and scalar curvature)**.** Let $\varphi\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be a local isometry between Riemannian manifolds. Then, for all $x \in M_1$, we have

$$\operatorname{scal}^{M_1,g_1}(x) = \operatorname{scal}^{M_2,g_2}\bigl(\varphi(x)\bigr).$$

This can be derived from the averaging description (Proposition 2.4.32) and the fact that local isometries preserve sectional curvature (Remark 2.4.21) and map orthonormal bases to orthonormal bases.

Proposition 2.4.34. *Homogeneous Riemannian manifolds have constant scalar curvature.*

Proof. This is a direct consequence of the invariance of scalar curvature under (local) isometries (Remark 2.4.33). □

Even though Ricci and scalar curvature are significantly weaker than sectional curvature, they often still contain enough information on (local) volume growth.

2.5 Model spaces

We now computate the sectional curvatures of the three types of model spaces: Euclidean spaces, spheres, hyperbolic spaces. For the computations, we will use the descriptions in terms of local coordinates. All model spaces have constant sectional curvature. We will later see that the model spaces are the only examples of simply connected Riemannian manifolds of constant sectional curvature.

To simplify the computation of curvatures of the model spaces, we will use two facts: The model spaces are locally conformally flat and they are highly symmetric.

2.5.1 Locally conformally flat manifolds

Locally conformally flat manifolds are obtained locally from flat manifolds by rescaling the Riemannian metric through a positive function. Using the local descriptions of the Riemannian curvature tensor, one can compute the curvature of locally conformally flat manifolds in terms of the rescaling function. The correction terms will involve several differentials, including the gradient.

Definition 2.5.1 (locally conformally flat manifold)**.**

- Let M be a smooth manifold. Riemannian metrics $\widetilde{g}$ and g on M are *conformal* if there exists a function $f \in C^\infty(M, \mathbb{R}_{>0})$ with

$$\widetilde{g} = f \cdot g.$$

- A Riemannian manifold (M, g) is *conformally flat* if g is conformal to a flat Riemannian metric on M.
- A Riemannian manifold (M, g) is *locally conformally flat* if each point of M admits an open neighbourhood that is conformally flat (with respect to the Riemannian metric induced by g).

Definition 2.5.2 (gradient). Let (M, g) be a Riemannian manifold and let $f \in C^\infty(M)$. Then the *gradient* of f is

$$\operatorname{grad} f := (\mathrm{d} f)^\sharp \in \Gamma(\mathrm{T}\, M).$$

Quick check 2.5.3 (gradient). Let (M, g) be a Riemannian manifold and let $f \in C^\infty(M)$. Then, for all $x \in M$ and all $v \in \mathrm{T}_x M$, we have

$$\big\langle (\operatorname{grad} f)(x), v \big\rangle_g = \boxed{?} \qquad .$$

Caveat 2.5.4. It should be noted that the gradient involves raising indices and thus depends on the underlying Riemannian metric! In order to avoid confusion, in Riemannian geometry, the gradient should *not* be denoted by ∇.

Theorem 2.5.5 (Riemannian curvature and conformal changes). *Let (M, g) be a Riemannian manifold of dimension n, let $f \in C^\infty(M, \mathbb{R}_{>0})$, and let $\widetilde{g} := e^{2f} \cdot g$. Let $\varphi\colon U \longrightarrow U'$ be a smooth chart for M, let $({\Gamma_{ij}}^k)_{i,j,k \in \{1,\dots,n\}}$ be the Christoffel symbols with respect to the coordinate frame $(E_1, \dots, E_n)$ associated with φ, and let $(g_{i,j})_{i,j \in \{1,\dots,n\}}$ denote the coefficients of g with respect to φ. The corresponding objects associated with $\widetilde{g}$ will be decorated with $\widetilde{\ }$. Then:*

1. *For all $i, j, k \in \{1, \dots, n\}$, we have*

$$\widetilde{\Gamma}_{ij}{}^k = {\Gamma_{ij}}^k + \delta_{jk} \cdot E_i(f) + \delta_{ik} \cdot E_j(f) - \sum_{\ell=1}^n g_{ij} \cdot g^{k\ell} \cdot E_\ell(f).$$

2. *For all $X, Y \in \Gamma(\mathrm{T}\, M)$, we have*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathrm{d} f(X) \cdot Y + X \cdot \mathrm{d} f(Y) - \langle X, Y \rangle_g \cdot \operatorname{grad} f.$$

3. *For all $i, j, k, \ell \in \{1, \dots, n\}$, we have*

$$\widetilde{R}_{ijk\ell} = e^{2f} \cdot R_{ijk\ell} - e^{2f} \cdot (g_{ik} \cdot T_{j\ell} + g_{j\ell} \cdot T_{ik} - g_{i\ell} \cdot T_{jk} - g_{jk} \cdot T_{i\ell})$$

where $T_{ij} := \nabla_{E_i} \nabla_{E_j} f - \nabla_{E_i} f \cdot \nabla_{E_j} f + \frac{1}{2} \cdot \|\operatorname{grad} f\|_g^2 \cdot g_{ij}$.

4. *If $(g_{ij})_{i,j \in \{1,\dots,n\}} = (\delta_{ij})_{i,j \in \{1,\dots,n\}}$, then for all $i, j \in \{1, \dots, n\}$, we obtain the simplified formula*

$$\begin{aligned}\widetilde{R}_{ijji} &= 0 - e^{2f} \cdot (T_{jj} - T_{ii}) \\ &= -e^{2f} \cdot \big(\nabla_{E_j} \nabla_{E_j} f - \nabla_{E_j} f \cdot \nabla_{E_j} f + \nabla_{E_i} \nabla_{E_i} f - \nabla_{E_i} f \cdot \nabla_{E_i} f + \|\operatorname{grad} f\|_g^2\big).\end{aligned}$$

Proof. This follows from the local descriptions in the fundamental theorem (proof of Theorem 2.3.17) and in Remark 2.4.5 (Exercise). $\square$

Our main examples of locally conformally flat manifolds will be the model spaces (Chapter 2.5.3).

2.5.2 Symmetries and constant curvature

The more symmetric Riemannian manifolds are, the more constant their curvatures are. In particular, it will turn out that the model spaces all have constant sectional curvature.

In Proposition 2.4.34, we already observed that homogeneous spaces have constant scalar curvature. For constant Ricci or constant sectional curvature, we need higher degrees of symmetry.

Quick check 2.5.6. It is clear what "constant scalar cuvature" and "constant sectional curvature" mean. Can you guess what "constant Ricci curvature" could mean?
Hint

Proposition 2.5.7. *Let (M, g) be a Riemannian manifold that is homogeneous and isotropic. Then (M, g) has constant Ricci curvature, i.e., there is a constant $c \in \mathbb{R}$ such that for all $x \in M$ and all $v \in \mathrm{T}_x M$ with $\|v\|_g = 1$, we have*

$$\mathrm{Ric}_x(v, v) = c.$$

Proof. Let $x, y \in M$ and let $v \in \mathrm{T}_x M$ and $w \in \mathrm{T}_y M$ with $\|v\|_g = 1 = \|w\|_g$. Because (M, g) is homogeneous and isotropic, there exists a $\varphi \in \mathrm{Isom}(M, g)$ with

$$\varphi(x) = y \quad \text{and} \quad \mathrm{d}_x \varphi(v) = w.$$

We use the averaging description of Ricci curvature (Proposition 2.4.28): Let $(v_1, \dots, v_n)$ be an orthonormal basis of $\mathrm{T}_x M$ (where $n := \dim M$) with $v_1 = v$. Then $(\mathrm{d}_x \varphi(v_1), \dots, \mathrm{d}_x \varphi(v_n))$ is an orthonormal basis of $\mathrm{T}_y(M)$ with $\mathrm{d}_x \varphi(v_1) = w$ and thus we obtain

$$\begin{aligned}
\mathrm{Ric}_x(v, v) &= \sum_{k=2}^{n} \sec_x(v_k, v_1) && \text{(Proposition 2.4.28)} \\
&= \sum_{k=2}^{n} \sec_y\bigl(\mathrm{d}_x \varphi(v_k), \mathrm{d}_x \varphi(v_1)\bigr) && \text{(Remark 2.4.21)} \\
&= \mathrm{Ric}_x(w, w), && \text{(Proposition 2.4.28)}
\end{aligned}$$

as claimed. $\square$

Definition 2.5.8 (2-isotropic manifold)**.** Let (M, g) be a Riemannian manifold.

- Let $x \in M$. Then, (M, g) is 2-*isotropic at* x, if the following holds: For all orthonormal pairs (v_1, v_2) and (v_1', v_2') of vectors in $\mathrm{T}_x M$, there exists an isometry $\varphi \in \mathrm{Isom}_x(M, g)$ with

$$\mathrm{d}_x \varphi(v_1) = v_1' \quad \text{and} \quad \mathrm{d}_x \varphi(v_2) = v_2'.$$

- Moreover, (M, g) is 2-*isotropic*, if (M, g) is 2-isotropic at every point.

Proposition 2.5.9. *Let (M, g) be a Riemannian manifold that is homogeneous and 2-isotropic. Then (M, g) has constant sectional curvature.*

Proof. Let $x, y \in M$ and let $v_1, v_2 \in \mathrm{T}_x M$, $w_1, w_2 \in \mathrm{T}_y M$ such that (v_1, v_2) is linearly independent and (w_1, w_2) is linearly independent. By Gram-Schmidt orthonormalisation, there exist $\widetilde{v}_1, \widetilde{v}_2 \in \mathrm{T}_x M$ and $\widetilde{w}_1, \widetilde{w}_2 \in \mathrm{T}_y M$ such that $(\widetilde{v}_1, \widetilde{v}_2)$ is an orthonormal basis of the plane $\mathrm{span}_{\mathbb{R}}(v_1, v_2) \subset \mathrm{T}_x M$ and such that $(\widetilde{w}_1, \widetilde{w}_2)$ is an orthonormal basis of the plane $\mathrm{span}_{\mathbb{R}}(w_1, w_2) \subset \mathrm{T}_y M$. Because (M, g) is homogeneous and 2-isotropic, there exists a $\varphi \in \mathrm{Isom}(M, g)$ with

$$\varphi(x) = y \quad \text{and} \quad \mathrm{d}_x \varphi(\widetilde{v}_1) = \widetilde{w}_1 \quad \text{and} \quad \mathrm{d}_x \varphi(\widetilde{v}_2) = \widetilde{w}_2$$

(by the same argument as in Proposition 1.4.4; check!). As sectional curvature is invariant under isometries and depends only on the corresponding tangent plane, we obtain

$$\begin{aligned}
\sec_x(v_1, v_2) &= \sec_x(\widetilde{v}_1, \widetilde{v}_2) && \text{(Proposition 2.4.22)}\\
&= \sec_{\varphi(x)}\bigl(\mathrm{d}_x \varphi(\widetilde{v}_1), \mathrm{d}_x \varphi(\widetilde{v}_2)\bigr) && \text{(Remark 2.4.21)}\\
&= \sec_y(\widetilde{w}_1, \widetilde{w}_2) && \text{(choice of } \varphi)\\
&= \sec_y(w_1, w_2). && \text{(Proposition 2.4.22)}
\end{aligned}$$

Thus, (M, g) has constant sectional curvature. □

Remark 2.5.10 (constant curvature)**.**

1. If a Riemannian manifold has constant sectional curvature, then also Ricci curvature and scalar curvature are constant (Proposition 2.4.28, Proposition 2.4.32).

   The converse is *not* true in general.

2. Riemannian quotients of Riemannian manifolds with constant sectional curvature by proper isometric group actions also have constant sectional curvature (Remark 2.4.21).

2.5.3 Sectional curvature of the model spaces

We are now well-prepared to compute the sectional curvature of the model spaces:

Theorem 2.5.11 (sectional curvature of the model spaces). *Let $n \in \mathbb{N}_{\geq 2}$ and let $R \in \mathbb{R}_{>0}$.*

1. *The Euclidean space $\mathbb{R}^n$ with the Euclidean Riemannian metric has constant sectional curvature 0.*
2. *The round sphere $\mathbb{S}^n(R)$ has constant sectional curvature $1/R^2$.*
3. *Hyperbolic space $\mathbb{H}^n(R)$ has constant sectional curvature $-1/R^2$.*

Quick check 2.5.12. In Theorem 2.5.11, why do we assume $n \geq 2$?
Because

Proof. In the Euclidean case, we already know this result (Example 2.4.20).

Spheres and hyperbolic spaces are homogeneous and isotropic (Proposition 1.4.10 and 1.4.12); in fact, the very same proof also shows that they are 2-isotropic. Thus, $\mathbb{S}^n(R)$ and $\mathbb{H}^n(R)$ both have constant sectional curvature (Proposition 2.5.9) and it suffices to compute the sectional curvature at a single point and for a single pair of linearly independent tangent vectors at this point.

For this computation, we use that spheres and hyperbolic spaces are locally conformally flat:

- For $\mathbb{H}^n(R)$, we use the description through the Poincaré disk model.
- For $\mathbb{S}^n(R)$, we use stereographic projection from the punctured sphere to $\mathbb{R}^n$ (Exercise).

In both cases, the conformal factor coincides up to a single sign; thus, we only need to consider the Riemannian metrics

$$g_{\pm} \colon x \longmapsto \frac{4 \cdot R^4}{\left(R^2 \pm \|x\|_2{}^2\right)^2} \cdot \overline{g}_x$$

on a small open neighbourhood U of 0 in $\mathbb{R}^n$, where $\overline{g}$ is the Euclidean Riemannian metric on $\mathbb{R}^n$. The sign "$-$" corresponds to the hyperbolic case, the sign "$+$" to the spherical case (Proposition 1.4.11; Exercise).

The goal is now to compute the single value $\sec_0^{U,g_{\pm}}(e_1, e_2)$, where e_1, e_2 are the first two standard unit normal vectors in $\mathbb{R}^n$. Using the identity chart on U (which leads to the standard coordinate frame $(E_1, \dots, E_n)$), we obtain from Theorem 2.5.5 that

$$\begin{aligned} R^{\pm}_{1221} = -e^{2\cdot f_\pm} \cdot \bigl(& \overline{\nabla}_{E_2}\overline{\nabla}_{E_2} f_\pm - \overline{\nabla}_{E_2} f_\pm \cdot \overline{\nabla}_{E_2} f_\pm \\ & + \overline{\nabla}_{E_1}\overline{\nabla}_{E_1} f_\pm - \overline{\nabla}_{E_1} f_\pm \cdot \overline{\nabla}_{E_1} f_\pm \\ & + \|\operatorname{grad} f_\pm\|_2^2 \bigr), \end{aligned}$$

where $\overline{\nabla}$ is the Euclidean linear connection on U and

$$\begin{aligned} f_\pm \colon U &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{1}{2} \cdot \ln \frac{4 \cdot R^4}{\bigl(R^2 \pm \|x\|_2^2\bigr)^2} = -\ln \frac{1}{4 \cdot R^2} \cdot \bigl(R^2 \pm \|x\|_2^2\bigr). \end{aligned}$$

It only remains to determine the corresponding covariant derivates. By definition of $\overline{\nabla}$, we have

$$\begin{aligned} (\overline{\nabla}_{E_j} f_\pm)(x) &= \mp 2 \cdot \frac{x_j}{R^2 \pm \|x\|^2} \\ (\overline{\nabla}_{E_j}\overline{\nabla}_{E_j} f_\pm)(x) &= \mp 2 \cdot \frac{R^2 \pm \|x\|_2^2 \mp 2 \cdot x_j^2}{\bigl(R^2 \pm \|x\|_2^2\bigr)^2} \end{aligned}$$

for all $j \in \{1, \dots, n\}$ and all $x \in U$. Therefore,

$$\bigl\|\operatorname{grad} f_\pm(x)\bigr\|_2^2 = 4 \cdot \frac{\|x\|_2^2}{\bigl(R^2 \pm \|x\|_2^2\bigr)^2}$$

and so

$$R^{\pm}_{1221}(0) = -4 \cdot \Bigl(\mp \frac{2 \cdot R^2}{R^4} - 0 \mp \frac{2 \cdot R^2}{R^4} - 0 + 4 \cdot 0\Bigr) = \pm \frac{16}{R^2}.$$

In total, we thus have

$$\sec_0^{U, g_\pm}(e_1, e_2) = \frac{R^{\pm}_{1221}(0)}{g_{\pm,11}(0) \cdot g_{\pm,22}(0) - 0} = \pm \frac{16}{R^2} \cdot \frac{R^4 \cdot R^4}{4 \cdot R^4 \cdot 4 \cdot R^4} = \pm \frac{1}{R^2},$$

as claimed. □

Study note (sectional curvature of hyperbolic spaces)**.** The easiest way to compute the sectional curvature of hyperbolic spaces is probably through the half-space model and Theorem 2.5.5: In this model, computations are fairly simple (you should do this computation!). The disadvantage is that this computation has no suitable counterpart in the spherical case. Therefore, we used the Poincaré disk model in the proof of Theorem 2.5.11.

Quick check 2.5.13 (Ricci and scalar curvature of the model spaces)**.** Using Theorem 2.5.11, one can compute also the Ricci and scalar curvatures of the model spaces. Let $n \in \mathbb{N}_{\geq 2}$ and let $R \in \mathbb{R}_{>0}$.

1. What is the Ricci curvature of Euclidean space?

Hint

2. What is the scalar curvature of Euclidean space?

   Hint

3. What is the Ricci curvature of $\mathbb{S}^n(R)$?

   Hint

4. What is the scalar curvature of $\mathbb{S}^n(R)$?

   Hint

5. What is the Ricci curvature of $\mathbb{H}^n(R)$?

   Hint

6. What is the Ricci curvature of $\mathbb{H}^n(R)$?

   Hint

All of these values are obtained immeditaley from Theorem 2.5.11, Proposition 2.4.28, and Proposition 2.4.32.

Example 2.5.14 (quotients of model spaces)**.** Because projections of isometric proper group actions are local isometries with respect to the quotient Riemannian metric (Proposition 1.4.20), we obtain from Theorem 2.5.11 and Remark 2.4.21:

- For all $n \in \mathbb{N}_{\geq 2}$, real projective space $\mathbb{R}P^n$ admits a Riemannian metric of constant sectional curvature 1.
- All lens spaces admit a Riemannian metric of constant sectional curvature 1.
- All closed surfaces of genus at least 2 admit a Riemannian metric of constant sectional curvature -1.

Example 2.5.15. The answers to all the questions on (local) isometry between the model sapces on page 72 are all negative (except for the obvious exceptions), because they all have different sectional curvature.

Study note. Write a summary of Chapter 2, keeping the following questions in mind:

- What is the geometric idea behind the definition of curvature?
- How can curvature be formalised?
- Which local descriptions of curvature exist?
- Which role is played by connections? Which connection is typically used on Riemannian manifolds? Why does it exist?
- Which different types of curvatures do you know?
- How can these curvatures be computed in concrete examples?

3

Riemannian geodesics

Geodesics are "straight" curves with respect to a given linear connection. We will now explore the metric properties of Riemannian geodesics, i.e., of geodesics for the Levi-Civita connection. More precisely, we will show that

- length-minimising curves are Riemannian geodesics and that
- Riemannian geodesics are locally length-minimising.

The key tools in working with geodesics are the exponential map (which is based on radial geodesics emanating from a given point) and variational techniques for the length functional.

Moreover, we will study variations of geodesics, so-called Jacobi fields. These are the source of many comparison results in Riemannian geometry. We will apply these results in Chapter 4.

In this chapter, on all Riemannian manifolds, we will always work with the Levi-Civita connection and all geodesics are geodesics with respect to the Levi-Civita connection.

Overview of this chapter.

Running example. The model spaces and their quotients, submanifolds of Euclidean space

3.1 The exponential map

The exponential map is a map from (an open neighbourhood of the zero-section of) the tangent bundle of a Riemannian manifold down to the manifold, given by following maximal radial geodesics (Figure 3.1). This map is convenient for many arguments in Riemannian geometry and leads to particularly nice coordinates, so-called normal coordinates.

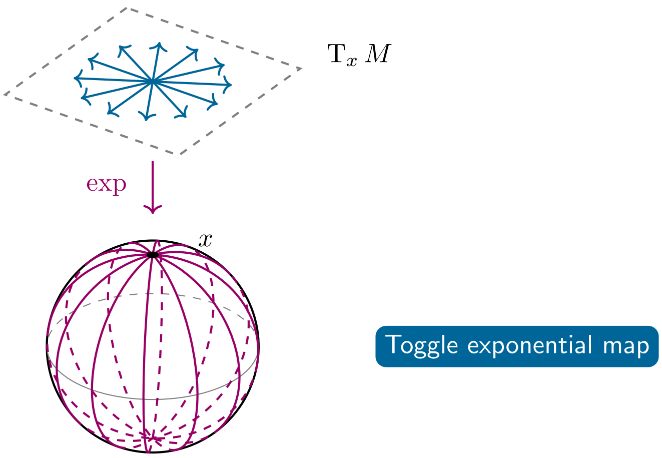


Figure 3.1.: The exponential map, schematically

3.1.1 The exponential map

Definition 3.1.1 (exponential map). Let (M, g) be a Riemannian manifold and let $x \in M$.

- We set $\mathrm{Exp} := \bigcup_{y \in M} \mathrm{Exp}_y M \subset \mathrm{T}\, M$, where

$$\mathrm{Exp}_x := \bigl\{ v \in \mathrm{T}_x M \bigm| \mathrm{geod}_{x,v} \text{ is defined on } [0,1] \bigr\} \subset \mathrm{T}_x M.$$

- The *exponential map of* (M, g) is given by

$$\begin{aligned} \exp\colon \mathrm{Exp} &\longrightarrow M \\ \mathrm{T}_x M \ni v &\longmapsto \mathrm{geod}_{x,v}(1). \end{aligned}$$

The restriction of exp to $\mathrm{T}_x M$ is denoted by $\exp_x \colon \operatorname{Exp}_x \longrightarrow M$.

If we want to emphasise the ambient manifold or Riemannian metric, we add corresponding superscripts.

Example 3.1.2 (exponential map of the circle)**.** The map (Figure 3.1)

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{S}^1 \\ t &\longmapsto (\cos t, \sin t) \text{ ``='' } \exp(i \cdot t) \end{aligned}$$

from Example 1.3.20 is the exponential map at $(1, 0)$ of $\mathbb{S}^1$, with respect to the round metric on $\mathbb{S}^1$ (check! Example 2.2.20).

Example 3.1.3 (exponential map in cartography)**.** The exponential map of $\mathbb{S}^2$ (with the Riemannian metric induced from the Euclidean Riemannian metric on $\mathbb{R}^3$) is the so-called azimuthal equidistant projection.

Example 3.1.4 (exponential map on the punctured plane)**.** On $M := \mathbb{R}^2 \setminus \{0\}$ (with the Riemannian metric induced by the Euclidean Riemannian metric on $\mathbb{R}^2$), is $\operatorname{Exp}_{(-1,0)} = \mathrm{T}_{(-1,0)} M$?

Yes No

Theorem 3.1.5 (exponential map: properties)**.** *Let (M, g) be a Riemannian manifold and let $x \in M$.*

1. *For all $v \in \mathrm{T}_x M$ and for all $t \in \mathbb{R}$, we have the following equality, whenever either side is defined:*

$$\operatorname{geod}_{x,v}(t) = \operatorname{geod}_{x,t \cdot v}(1).$$

2. *The set $\operatorname{Exp} \subset \mathrm{T} M$ is open and $\operatorname{Exp}_x \subset \mathrm{T}_x M$ is star-shaped (with respect to 0).*

3. *The map $\exp \colon \mathrm{T} M \longrightarrow M$ is smooth and under the canonical identification of $\mathrm{T}_0(\mathrm{T}_x M)$ with $\mathrm{T}_x M$ we have*

$$\mathrm{d}_0 \exp_x = \mathrm{id}_{\mathrm{T}_x M} .$$

4. *There exists an open neighbourhood $U \subset \mathrm{T}_x M$ of 0 such that $\exp_x(U) \subset M$ is open and $\exp_x|_U \colon U \longrightarrow \exp_x(U)$ is a diffeomorphism.*

Proof. Ad 1. This follows from the defining properties of the geodesics $\operatorname{geod}_{x,\dots}$ and basic differentiation rules (Exercise).

Ad 2./3. From the relation in the first part, it is easy to see that Exp_x is star-shaped with respect to 0 (check!). The fact that Exp is open and exp is smooth follows from the smooth dependence of solutions to the geodesic equation (Remark 2.2.21) on the initial conditions [15, Theorem I.6.1][18, Proposition 5.19].

We compute the differential of $\exp_x$ at 0: Let $v \in \mathrm{T}_0(\mathrm{T}_x M)$. Then, v is represented by the smooth curve $\gamma \colon t \longmapsto t \cdot v$ in $\mathrm{T}_x M$. Therefore,

$$\begin{aligned}
(\mathrm{d}_0 \exp_x)(v) &= [\exp_x \circ \gamma] \\
&= \bigl[t \longmapsto \mathrm{geod}_{x,t\cdot v}(1)\bigr] && \text{(by definition of } \gamma \text{ and } \exp_x\text{)} \\
&= [\mathrm{geod}_{x,v}] && \text{(by the first part)} \\
&= v. && \text{(by the definition of } \mathrm{geod}_{x,v}\text{)}
\end{aligned}$$

Ad 4. This is a local problem. By the third part, $\exp_x$ is smooth and $\mathrm{d}_0 \exp_x$ is invertible. Therefore, the inverse function theorem finishes the proof. □

Quick check 3.1.6. In the situation of Theorem 3.1.5, is the exponential map even an isometry in a small neighbourhood of $0 \in \mathrm{T}_x M$ (with respect to g_x on $\mathrm{T}_x M$)?

Yes No

Proposition 3.1.7 (naturality of the exponential map)**.** *Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds and let $\varphi \colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be an isometry.*

1. *If γ is a geodesic of (M_1, g_1), then $\varphi \circ \gamma$ is a geodesic of (M_2, g_2).*
2. *For all $x \in M_1$ and all $v \in \mathrm{Exp}_x$, we have*

$$\exp^{M_2,g_2}_{\varphi(x)}\bigl(\mathrm{d}_x \varphi(v)\bigr) = \varphi\bigl(\exp^{M_1,g_1}_x(v)\bigr).$$

$$\begin{array}{ccc}
\mathrm{Exp}^{M_1,g_1} & \xrightarrow{\mathrm{d}\varphi} & \mathrm{Exp}^{M_2,g_2} \\
{\scriptstyle \exp^{M_1,g_1}}\downarrow & & \downarrow{\scriptstyle \exp^{M_2,g_2}} \\
M_1 & \xrightarrow[\varphi]{} & M_2
\end{array}$$

Proof. Ad 1. This follows from the compatibility of covariant derivatives along curves with the connection and the naturality of the Levi-Civita connection (Proposition 2.3.20; check!).

Ad 2. Using the first part, we obtain

$$\begin{aligned}
\exp^{M_2,g_2}_{\varphi(x)}\bigl(\mathrm{d}_x \varphi(v)\bigr) &= \mathrm{geod}^{M_2,g_2}_{\varphi(x),\mathrm{d}_x \varphi(v)}(1) \\
&= \varphi\bigl(\mathrm{geod}^{M_1,g_1}_{x,v}(1)\bigr) && \text{(by the first part and uniqueness of geodesics)} \\
&= \varphi\bigl(\exp^{M_1,g_1}_x(v)\bigr),
\end{aligned}$$

as claimed. □

Remark 3.1.8 (naturality of the exponential map, local isometries)**.** There is also a corresponding version of Proposition 3.1.7 for local isometries: All relevant terms are local (check!).

3.1.2 Normal coordinates

Definition 3.1.9 (normal neighbourhood, normal coordinates)**.** Let (M, g) be a Riemannian manifold and let $x \in M$.

- An open neighbourhood U of x in M is *normal* if it is the diffeomorphic image of a star-shaped (with respect to 0) open neighbourhood of $0 \in \mathrm{T}_x M$ under the exponential map $\exp_x$.
- Let $U \subset M$ be a normal neighbourhood of x, let $(v_1, \dots, v_n)$ be an orthonormal basis of $\mathrm{T}_x M$, and let

  $$\begin{aligned} E \colon \mathbb{R}^n &\longrightarrow \mathrm{T}_x M \\ x &\longmapsto \sum_{j=1}^n x^j \cdot v_j \end{aligned}$$

  be the associated $\mathbb{R}$-linear isomorphism. Then the smooth chart

  $$E^{-1} \circ \exp_x^{-1}|_U : U \longrightarrow \mathbb{R}^n$$

  is called a *normal coordinate chart* around x.

Remark 3.1.10 (existence of normal coordinates)**.** In a Riemannian manifold, every point admits a normal neighbourhood (by Theorem 3.1.5) and hence, around every point, there exist normal coordinates.

Definition 3.1.11 (geodesic ball)**.** Let (M, g) be a Riemannian manifold, let $x \in M$, and let $r \in \mathbb{R}_{>0}$ with $B_r^{\mathrm{T}_x M, g_x}(0) \subset \mathrm{Exp}_x$. Then

$$\mathrm{ball}_r^{M,g}(x) := \exp_x\bigl(B_r^{\mathrm{T}_x M, g_x}(0)\bigr) \subset M$$

is the *geodesic ball of radius r around x*. Analogously, one defines *closed geodesic balls* and *geodesic spheres* $\mathrm{sphere}_r^{M,g}(x)$.

Caveat 3.1.12. Until now, we do not know how geodesic balls are related to metric balls (with respect to the metric induced by the Riemannian metric). This will be cleared up in the following sections.

Quick check 3.1.13. Let (M, g) be a Riemannian manifold and let $x \in M$. Is then every geodesic ball around x homeomorphic to a Euclidean ball?
Yes No

Proposition 3.1.14 (uniformly normal neighbourhood). *Let (M, g) be a Riemannian manifold and let $x \in M$. Then x admits a* uniformly normal neighbourhood*, i.e., an open neighbourhood U with the following property: There exists an $r \in \mathbb{R}_{>0}$ such that for each $y \in U$, the geodesic ball around y is a normal neighbourhood of y and contains U.*

Proof. This is an exercise in compactness and in unfolding quantifiers in the correct order [18, Theorem 5.25]. □

As a first example application of the existence of normal neighbourhoods, we establish that isometries are uniquely determined by the value and the differential at a single point.

Proposition 3.1.15 (initial data for local isometries). *Let (M, g) be a connected Riemannian manifold and let $\varphi, \psi \in \operatorname{Isom}(M, g)$. Moreover, let $x \in M$ with*

$$\varphi(x) = \psi(x) \quad \textit{and} \quad \mathrm{d}_x \varphi = \mathrm{d}_x \psi.$$

Then $\varphi = \psi$.

Proof. Looking at $\varphi \circ \psi^{-1}$, we only need to consider the case that $\psi = \mathrm{id}_M$ and that $\varphi(x) = x$, $\mathrm{d}_x \varphi = \mathrm{id}_{\mathrm{T}_x M}$. We need to show that, in this situation, $\varphi = \mathrm{id}_M$.

- We first prove the claim in the special case that all of M is a normal neighbourhood of x. Then, using the naturality of the exponential map (Proposition 3.1.7), we obtain: Let $y \in M$, say $y = \exp_x(v)$ with $v \in \mathrm{T}_x M$ (normal coordinates!). Then

$$\begin{aligned} \varphi(y) &= \varphi\bigl(\exp_x(v)\bigr) \\ &= \exp_{\varphi(x)}\bigl(\mathrm{d}_x \varphi(v)\bigr) && \text{(naturality of exp)} \\ &= \exp_{\varphi(x)}(v) && \text{(because } \mathrm{d}_x \varphi = \mathrm{id}_{\mathrm{T}_x M}) \\ &= \exp_x(v) && \text{(because } \varphi(x) = x) \\ &= y. \end{aligned}$$

- For the general case, we argue as follows: Let $y \in M$. Because M is connected (whence, as a manifold, path-connected), there exists a continuous path γ connecting x and y. In view of the existence of normal coordinates (Remark 3.1.10), we can cover the image of γ by normal neighbourhoods. As the image of γ is compact, finitely many of these suffice. Therefore, we can successively apply the special case above and obtain $\varphi(y) = y$. □

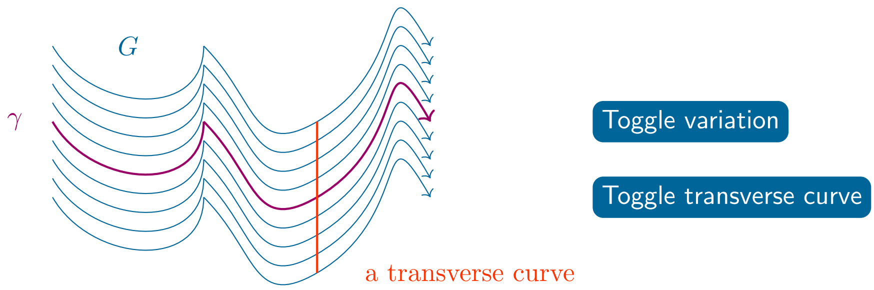


Figure 3.2.: Variation of a curve, schematically

3.2 Riemannian geodesics

We will now explore the metric properties of Riemannian geodesics. The goal is to establish a close relation between (locally) length-minimising curves and Riemannian geodesics.

Length-minimising curves are curves that minimise the length functional. In order to avoid technicalities by working on the infinite-dimensional space of curves, we only look the length functional on variations of curves, i.e., on one-parameter families of curves.

We will now work out the details of this approach.

3.2.1 Variation of curves

A variation of a curve is a one-parameter family of curves around this curve (Figure 3.2). As in the definition of length and distance, we will impose some regularity conditions on such families.

To solve the minimisation problem for the length functional, we need to compute the first derivative of the length functional on variations; this leads to the first variation formula.

Definition 3.2.1 (piecewise regular family)**.** Let M be a smooth manifold, let $a, b \in \mathbb{R}$ with $a < b$, and let $\varepsilon \in \mathbb{R}_{>0}$.

- A map $G \colon (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$ is *piecewise smooth* if G is continuous and there exists a partition $a = a_0 < \cdots < a_k = b$ of $[a, b]$ such that for each $j \in \{1, \ldots, k\}$, the restriction $G|_{(-\varepsilon,\varepsilon)\times[a_{j-1},a_j]}$ is smooth.

- A piecewise smooth map $G\colon (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$ is a *piecewise regular family (of curves)*, if for each $s \in (-\varepsilon,\varepsilon)$, the map $G(s,\,\cdot\,)\colon [a,b] \longrightarrow M$ is a piecewise regular curve.
- If $G\colon (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$ is a piecewise regular family, then for each $t \in [a,b]$, the map $G(\,\cdot\,,t)$ is the *transverse curve of G at t*.
- Let $G\colon (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$ be a piecewise regular family. A *vector field along G* is a piecewise smooth map $V\colon (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow \mathrm{T}M$ with
$$\forall_{(s,t)\in(-\varepsilon,\varepsilon)\times[a,b]} \quad V(s,t) \in \mathrm{T}_{G(s,t)}M.$$
(The map V is allowed to require a finer partition of $[a,b]$ than G !)

Quick check 3.2.2 (piecewise regular families of curves)**.** Let $G\colon (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$ be a piecewise regular family of curves.

1. Is $G(s,\,\cdot\,)$ for each $s \in (-\varepsilon,\varepsilon)$ smooth?

   Yes No

2. Is the transverse curve $G(\,\cdot\,,t)$ for each $t \in [a,b]$ smooth?

   Yes No

Remark 3.2.3 (vector fields associated with a piecewise regular family)**.** Let M be a smooth manifold and let $G\colon (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$ be a piecewise regular family of curves on M. We then write

$$\begin{aligned}\partial_{②}G(s,t) &:= \bigl(G(s,\,\cdot\,)\bigr)'(t)\\ \partial_{①}G(s,t) &:= \bigl(G(\,\cdot\,,t)\bigr)'(s),\end{aligned}$$

wherever these expressions are defined.

It should be noted that $\partial_{①}G$ is defined on all of $(-\varepsilon,\varepsilon)\times[a,b]$ and that $\partial_{①}G$ is continuous (check!). Therefore, $\partial_{①}G$ defines a vector field along G.

In contrast, $\partial_{②}G$ is not necessarily everywhere defined (this depends on the admissible partitions for G).

Proposition 3.2.4 (symmetry lemma for variations)**.** *Let (M,g) be a Riemannian manifold and let $G\colon (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$ be a piecewise regular family. Then, on every rectangle $Q = (-\varepsilon,\varepsilon)\times[\widetilde{a},\widetilde{b}] \subset (-\varepsilon,\varepsilon)\times[a,b]$ on which G is smooth, we have*

$$\forall_{(s,t)\in Q} \quad \bigl(D_{G(\,\cdot\,,t)}\partial_{②}G\bigr)(s,t) = \bigl(D_{G(s,\,\cdot\,)}\partial_{①}G\bigr)(s,t).$$

Proof. This is a consequence of the symmetry of the Levi-Civita connection on (M,g).

We first observe that this is a local problem; we thus compute both sides in local coordinates of M in an open neighbourhood of a given point in $G(Q)$ with respect to the corresponding coordinate frame $(E_1, \dots, E_n)$. Let $x^1, \dots, x^n \colon U \longrightarrow \mathbb{R}$ be the associated coordinate functions of G. Then, on suitable small enough open subsets U of Q,

$$\partial_{②} G = \sum_{k=1}^{n} \partial_2 x^k \cdot E_k$$
$$\partial_{①} G = \sum_{k=1}^{n} \partial_1 x^k \cdot E_k.$$

Using the local descriptions of the covariant derivative (proof of Theorem 2.2.16), we obtain

$$(D_{G(\,\cdot\,,t)} \partial_{②} G)(s,t) = \sum_{k=1}^{n} \partial_1 \partial_2 x^k(s,t) \cdot E_k(G(s,t)) + \sum_{i,j,k \in \{1,\dots,n\}} \partial_2 x^i(s,t) \cdot \partial_1 x^j(s,t) \cdot {\Gamma_{ji}}^k(G(s,t))$$
$$(D_{G(s,\,\cdot\,)} \partial_{①} G)(s,t) = \sum_{k=1}^{n} \partial_2 \partial_1 x^k(s,t) \cdot E_k(G(s,t)) + \sum_{i,j,k \in \{1,\dots,n\}} \partial_1 x^i(s,t) \cdot \partial_2 x^j(s,t) \cdot {\Gamma_{ji}}^k(G(s,t))$$

for all $(s,t) \in U$. The first sums coincide because the $x^1, \dots, x^n$ are smooth. Because the Levi-Civita connection is symmetric, the Christoffel symbols are symmetric (Proposition 2.3.12). Therefore, also the second sums coincide. $\square$

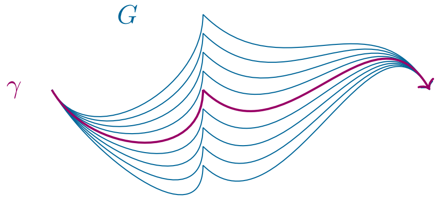
Definition 3.2.5 ((proper) variation of a curve)**.** Let M be a smooth manifold and let $\gamma \colon [a,b] \longrightarrow M$ be a piecewise regular curve.

- A *variation of* γ is a piecewise regular family $G \colon (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow M$ (for some $\varepsilon \in \mathbb{R}_{>0}$) such that
$$G(0,\,\cdot\,) = \gamma.$$
- A variation G of γ is a *proper variation of* γ, if (Figure 3.3)
$$\forall_{s \in (-\varepsilon,\varepsilon) \times [a,b]} \quad G(s,a) = \gamma(a) \land G(s,b) = \gamma(b).$$

3.2.2 Variation fields and the first variation formula

We now study variation fields, i.e., “vertical” derivatives of variations of curves. The key result is the first variation formula.

Definition 3.2.6 (variation field)**.** Let M be a smooth manifold, let $\gamma \colon [a,b] \longrightarrow M$ be a piecewise regular curve in M, and let G be a variation of γ. The *variation field of* G is the vector field



Toggle variation

Figure 3.3.: Proper variation of a curve, schematically

$$\begin{aligned} [a,b] &\longrightarrow \mathrm{T}\,M \\ t &\longmapsto \partial_{①} G(0,t) \end{aligned}$$

along γ.

Proposition 3.2.7 (existence of variation fields)**.** *Let M be a smooth manifold, let $\gamma\colon [a,b] \longrightarrow M$ be a piecewise regular curve in M, and let V be a vector field along γ. Then V is the variation field of some variation of γ.*

If V is proper *(i.e., if $V(a) = 0 = V(b)$), then we can find such a variation that is proper.*

Proof. Let g be a Riemannian metric on M (Theorem 1.3.6) The idea is to set

$$G(s,t) := \exp_{\gamma(t)}\bigl(s \cdot V(t)\bigr).$$

What is the problem with this definition? Hint

Because $[a,b]$ is compact and the domain of the exponential map is open (Theorem 3.1.5), there indeed exists an $\varepsilon \in \mathbb{R}_{>0}$ such that G is defined on $(-\varepsilon, \varepsilon) \times [a,b]$ (check!)

Let $a = a_0 < \cdots < a_k = b$ be a subdivision of $[a,b]$ such that V is smooth on all $[a_{j-1}, a_j]$. Then, G is continuous (check!) and on each rectangle $(-\varepsilon, \varepsilon) \times [a_{j-1}, a_j]$, the map G is smooth (check!). Moreover, $G(0, \,\cdot\,) = \exp_{\gamma(\,\cdot\,)}(0) = \gamma$.

By construction and the basic properties of the exponential map (Theorem 3.1.5), for all $t \in [a,b]$, we have

$$\partial_{①} G(0,t) = \boxed{?} \qquad = 1 \cdot V(t).$$

Therefore, V is the variation field of G.

Furthermore, if V is proper, then G is a proper variation of γ, because $G(\,\cdot\,, a) = \exp_{\gamma(a)}(0) = \gamma(a)$ and $G(\,\cdot\,, b) = \gamma(b)$. $\square$

Theorem 3.2.8 (the first variation formula)**.** *Let (M,g) be a Riemannian manifold, let $\gamma\colon [a,b] \longrightarrow M$ be a unit speed piecewise regular curve, let*

$G\colon (-\varepsilon,\varepsilon)\times[a,b]\longrightarrow M$ be a proper variation of γ, and let V be the variation field of G. Moreover, let $a=a_0<\dots<a_k=b$ be a partition such that $\gamma|_{[a_{j-1},a_j]}$ is smooth for all $j\in\{1,\dots,k\}$, and let $\Delta_j\dot\gamma := \dot\gamma(a_j^+)-\dot\gamma(a_j^-)$ denote the jump of $\dot\gamma$ at a_j. Then

$$\begin{aligned}\frac{\partial}{\partial s}\Big|_{s=0} L_g\big(G(s,\,\cdot\,)\big) = &-\int_a^b \big\langle V(t), D_\gamma\dot\gamma(t)\big\rangle_g\,dt && \textit{(regular contribution)}\\ &-\sum_{j=1}^{k-1}\big\langle V(a_j),\Delta_j\dot\gamma\big\rangle_g. && \textit{(singular contribution)}\end{aligned}$$

Proof. We compute the pieces $\big(s\mapsto L_g\big(G(s,\,\cdot\,)|_{[a_{j-1},a_j]}\big)\big)'(0)$ individually and then sum up all these contributions. During these computations, we use the abbreviations (wherever they are defined)

$$\begin{aligned} T\colon (s,t)&\longmapsto \partial_{②}G(s,t)\\ S\colon (s,t)&\longmapsto \partial_{①}G(s,t).\end{aligned}$$

Let $j\in\{1,\dots,k\}$. Then, we have

$$\begin{aligned}
&\frac{\partial}{\partial s}\Big|_{s=0} L_g\big(G(s,\,\cdot\,)|_{[a_{j-1},a_j]}\big)\\
&= \frac{\partial}{\partial s}\Big|_{s=0}\int_{a_{j-1}}^{a_j}\big\langle T(s,t),T(s,t)\big\rangle_g^{1/2}\,dt && \text{(definition of } L_g \text{ and } T)\\
&= \int_{a_{j-1}}^{a_j}\frac{\partial}{\partial s}\Big|_{s=0}\big\langle T(s,t),T(s,t)\big\rangle_g^{1/2}\,dt && \text{(elementary analysis)}\\
&= \int_{a_{j-1}}^{a_j}\frac{1}{2}\cdot\big\langle T(0,t),T(0,t)\big\rangle^{-1/2}\cdot 2\cdot\big\langle D_{G(\,\cdot\,,t)}T(0,t),T(0,t)\big\rangle\,dt && \text{(chain rule; Proposition 2.3.7)}\\
&= \int_{a_{j-1}}^{a_j}\frac{1}{\|T(0,t)\|_g}\cdot\big\langle D_{G(0,\,\cdot\,)}S(0,t),T(0,t)\big\rangle\,dt. && \text{(symmetry lemma: Proposition 3.2.4)}\\
&= \int_{a_{j-1}}^{a_j}\frac{1}{\|\dot\gamma(t)\|_g}\cdot\big\langle D_\gamma V(t),\dot\gamma(t)\big\rangle_g\,dt && \text{(rewriting via } \gamma \text{ and } V)\\
&= \int_{a_{j-1}}^{a_j}\big\langle D_\gamma V(t),\dot\gamma(t)\big\rangle_g\,dt && (G(0,\,\cdot\,)=\gamma \text{ has unit speed})\\
&= \int_{a_{j-1}}^{a_j}\big((\langle V,\dot\gamma\rangle_g)' - \langle V, D_\gamma\dot\gamma\rangle_g\big) && \text{(Proposition 2.3.7)}\\
&= \big\langle V(a_j),\dot\gamma(a_j^-)\big\rangle_g - \big\langle V(a_{j-1}),\dot\gamma(a_{j-1}^+)\big\rangle_g - \int_{a_{j-1}}^{a_j}\big\langle V(t),D_\gamma\dot\gamma(t)\big\rangle_g\,dt && \text{(fundamental theorem of calculus)}
\end{aligned}$$

Summing all these contributions and noting that $V(a_0)=0=V(a_k)$ (because) completes the proof of the first variation formula. □

3.2.3 Minimising curves are geodesics

The first variation formula allows us to show that minimising curves are geodesics. More precisely, we first relate geodesics to critical points of the length functional; in a second step, we show that minimising curves are critical points.

To define the notion of critical points, we use variations – so that we can work with elementary analysis instead of analysis on the infinite-dimensional space of all curves.

Definition 3.2.9 (minimising curve)**.** Let (M, g) be a Riemannian manifold. A piecewise regular curve $\gamma\colon [a, b] \longrightarrow M$ is *minimising*, if all piecewise regular curves η in M with the same endpoints satisfy

$$L_g(\eta) \geq L_g(\gamma).$$

Definition 3.2.10 (critical point)**.** Let (M, g) be a Riemannian manifold. A piecewise regular unit speed curve γ on M is *a critical point of L_g*, if for all proper variations G of γ, we have

$$\frac{\partial}{\partial s}\Big|_{s=0} L_g\big(G(s,\ \cdot\ )\big) = 0.$$

Theorem 3.2.11 (critical points vs. geodesics)**.** *Let (M, g) be a Riemannian manifold and let γ be a piecewise regular unit speed curve on M. Then the following are equivalent:*

1. *The curve γ is a critical point of L_g.*
2. *The curve γ is a geodesic.*

Proof. We write $[a, b]$ for the domain of γ.

Ad 1. $\Longrightarrow$ 2. Let γ be a critical point of L_g. We show that γ is a geodesic, i.e., that γ is smooth and that $D_\gamma \dot\gamma = 0$. We achieve this goal as follows:

① We show that the geodesic property holds on all smooth points of γ.

② We show that γ is smooth.

Both these properties will be extracted from the first variation formula (Theorem 3.2.8).

Ad ①. Let $[\widetilde{a}, \widetilde{b}] \subset [a, b]$ be a sub-interval with non-empty interior on which γ is smooth. We localise the problem via a bump function; thus, let $\varphi\colon [a, b] \longrightarrow \mathbb{R}$ be a smooth function with

$$\varphi|_{(\widetilde{a},\widetilde{b})} > 0 \quad \text{and} \quad \varphi|_{[a,b]\setminus(\widetilde{a},\widetilde{b})} = 0.$$

We consider the truncated vector field $V := \varphi \cdot D_\gamma \dot\gamma$ along γ. By Proposition 3.2.7, there exists a proper variation G of γ with variation field V. Because γ is a critical point of L_g, we obtain from the first variation formula that

$$\begin{aligned}
0 &= -\int_a^b \langle V, D_\gamma\dot\gamma\rangle_g - \sum_{i=1}^{k-1} \bigl\langle V(a_i), \Delta_i\dot\gamma\bigr\rangle_g \\
&= -\int_a^b \varphi \cdot \langle D_\gamma\dot\gamma, D_\gamma\dot\gamma\rangle_g - \sum_{j=1}^{k-1} \varphi(a_j) \cdot \bigl\langle D_\gamma\dot\gamma(a_j), \Delta_j\dot\gamma\bigr\rangle_g \\
&= -\int_{\widetilde a}^{\widetilde b} \varphi(t) \cdot \bigl\|D_\gamma\dot\gamma(t)\bigr\|_g^2 \, dt - 0. \qquad \text{(vanishing properties of } \varphi)
\end{aligned}$$

Because the integrand is continuous and φ is positive on $(\widetilde a, \widetilde b)$, we obtain

$$\forall_{t \in (\widetilde a, \widetilde b)} \quad D_\gamma\dot\gamma(t) = 0.$$

Ad ②. It suffices to show that $\Delta_j\dot\gamma = 0$ for all $j \in \{1, \dots, k-1\}$. Let $j \in \{1, \dots, k-1\}$. Using a bump function, we can find a continuous vector field V along γ with

$$V(a_j) = \Delta_j\dot\gamma \quad \text{and} \quad \forall_{i \in \{1,\dots,k-1\}\setminus\{j\}} \quad V(a_i) = 0.$$

Then, there exists a proper variation field G of γ, whose variation field is V (Proposition 3.2.7). Therefore, the first variation formula gives us

$$\begin{aligned}
0 &= -\int_a^b \langle V, D_\gamma\dot\gamma\rangle_g - \sum_{j=1}^{k-1} \bigl\langle V(a_j), \Delta_j\dot\gamma\bigr\rangle_g \\
&= -0 - \langle \Delta_j\dot\gamma, \Delta_j\dot\gamma\rangle_g \qquad \text{(by ① and the choice of } V) \\
&= -\|\Delta_j\dot\gamma\|_g^2,
\end{aligned}$$

and hence $\Delta_j\dot\gamma = 0$.

Ad 2. $\Longrightarrow$ 1. Conversely, now let γ be a geodesic. We show that γ is a critical point of L_g. Thus, let $G\colon (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$ be a proper variation of γ. Then, the first variation formula (Theorem 3.2.8) yields

$$\frac{\partial}{\partial s}\Big|_{s=0} L_g\bigl(G(s, \,\cdot\,)\bigr) = 0,$$

because: The singular contribution vanishes because ; the regular contribution vanishes because . ☐

Corollary 3.2.12 (minimising curves are geodesics). *Let (M, g) be a Riemannian manifold. Every minimising curve (with unit speed parametrisation) on (M, g) is a Riemannian geodesic of (M, g) (and, in particular, smooth).*

Proof. Let $\gamma\colon [a, b] \longrightarrow M$ be a minimising curve with unit speed parametrisation. In view of Theorem 3.2.11, it suffices to show that γ is a critical point of L_g.

Let $G\colon (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$ be a proper variation of γ; in particular, for each $s \in (-\varepsilon, \varepsilon)$, the curve $G(s, \cdot\,)$ is a piecewise regular curve from $\gamma(a)$ to $\gamma(b)$. Because $s \longmapsto L_g(G(s, \cdot\,))$ is smooth (check!) and γ is assumed to be minimising, we obtain

$$\frac{\partial}{\partial s}\Big|_{s=0} L_g\big(G(s, \cdot\,)\big) = 0.$$

Hence, γ is a critical point of L_g. □

3.2.4 Geodesics are locally minimising

Conversely, we complete the picture by showing that Riemannian geodesics are locally minimising.

As this is a local statement, the idea is to prove this claim only for radial geodesics in (uniformly) normal neighbourhoods. For radial geodesics, the proof is based on orthogonality of the geodesic spheres and the unit radial vector field (a fact known as the Gauß lemma). We first outline the main proof, and then establish the ingredients going into this proof.

Theorem 3.2.13. *Every geodesic on a Riemannian manifold has constant speed and locally is a minimising curve.*

Proof. Let (M, g) be a Riemannian manifold and let $\gamma\colon I \longrightarrow M$ be a geodesic; without loss of generality, we may assume that I is an open interval (check!). It is a simple observation that γ has constant speed (Proposition 3.2.14 below). Let $t \in I$, let U be a uniformly normal neighbourhood of $x := \gamma(t)$ (Proposition 3.1.14), and let $J \subset \gamma^{-1}(U)$ be the connected component that contains t.

Let $a, b \in J$ with $a < t < b$. We show that $\gamma|_{[a,b]}$ is a minimising curve: Because U is a uniformly normal neighbourhood, $\gamma(b)$ is contained in a geodesic ball around $\gamma(a)$. Therefore, the radial geodesic from $\gamma(a)$ to $\gamma(b)$ is simultaneously

- the unique (up to affine reparametrisation) geodesic in U between them (Proposition 3.2.15), and
- a minimising curve between them (Corollary 3.2.21).

Thus, $\gamma|_{(a,b)}$ must be this radial geodesic (up to affine reparametrisation), whence also minimising. □

Proposition 3.2.14. *Geodesics in Riemannian manifolds have constant speed.*

Proof. Let (M, g) be a Riemannian manifold and let $\gamma\colon I \longrightarrow M$ be a geodesic. Then, we obtain for all $t \in I^\circ$:

$$\begin{aligned}
\bigl(\|\dot\gamma\|_g^2\bigr)'(t) &= \bigl\langle \dot\gamma, \dot\gamma \bigr\rangle_g'(t) \\
&= 2 \cdot \bigl\langle D_\gamma \dot\gamma(t), D_\gamma \dot\gamma(t) \bigr\rangle_g && (\text{by} \qquad) \\
&= 2 \cdot \langle 0, 0 \rangle_g && (\gamma \text{ is a geodesic}) \\
&= 0.
\end{aligned}$$

Therefore, $\|\dot\gamma\|_g$ is constant, i.e., γ has constant speed. □

Proposition 3.2.15 (uniqueness of radial geodesics as geodesics)**.** *Let (M, g) be a Riemannian manifold, let $x \in M$, let $U \subset M$ be a normal neighbourhood of x, let $y \in U \setminus \{x\}$, and let $\gamma\colon [a, b] \longrightarrow U$ be a geodesic with $\gamma(a) = x$ and $\gamma(b) = y$. Then,*

$$\forall_{t \in [a,b]} \quad \gamma(t) = \operatorname{geod}_{x, \exp_x^{-1}(y)}\Bigl(\frac{t-a}{b-a}\Bigr).$$

Proof. In view of uniqueness of geodesics, we know that

$$\forall_{t \in [a,b]} \quad \gamma(t) = \operatorname{geod}_{x,v}(t-a),$$

where $v := \dot\gamma(a)$. We now relate v to $\exp_x^{-1}(y)$. From the previous description of γ, we obtain

$$\begin{aligned}
y = \gamma(b) &= \operatorname{geod}_{x,v}(b-a) = \operatorname{geod}_{x,(b-a)\cdot v}(1) && (\text{Theorem 3.1.5}) \\
&= \exp_x\bigl((b-a)\cdot v\bigr),
\end{aligned}$$

and so $v = 1/(b-a) \cdot \exp_x^{-1}(y)$. Therefore, for all $t \in [a, b]$, we have

$$\begin{aligned}
\gamma(t) &= \operatorname{geod}_{x,v}(t-a) = \operatorname{geod}_{x,1/(b-a)\cdot\exp_x^{-1}(y)}(t-a) \\
&= \operatorname{geod}_{x,\exp_x^{-1}(y)}\Bigl(\frac{t-a}{b-a}\Bigr) && (\text{Theorem 3.1.5}),
\end{aligned}$$

as desired. □

Quick check 3.2.16. In the proof of Proposition 3.2.15, where did we use that assumption on U?

When

To complete the proof of Theorem 3.2.13, we still need to establish that radial geodesics are minimising. Radial geodesics are integral curves of the radial vector field and the Gauß lemma shows that the radial vector field is orthogonal to the geodesic spheres.

Definition 3.2.17 (unit radial vector field)**.** Let (M, g) be a Riemannian manifold, let $x \in M$, and let $U \subset M$ be a normal neighbourhood of x.

- The *radial distance function at x* is defined by

$$\begin{aligned} \varrho \colon U &\longrightarrow \mathbb{R}_{\geq 0} \\ y &\longmapsto \left\| \exp_x^{-1}(y) \right\|_g . \end{aligned}$$

- The *unit radial vector field at x* is the vector field on $U \setminus \{y\}$ defined by

$$\begin{aligned} \frac{\partial}{\partial \varrho} \colon U \setminus \{x\} &\longrightarrow \mathrm{T}\, M \\ y &\longmapsto \frac{1}{\varrho(y)} \cdot \dot\gamma(1), \text{ where } w := \exp_x^{-1}(y) \text{ and } \gamma := \operatorname{geod}_{x,w}. \end{aligned}$$

Remark 3.2.18 (on the definition of the unit radial vector field)**.** In the situation of Definition 3.2.17, why is it justified to call $\partial/\partial\varrho$ the *unit* radial vector field?

Indeed, for all $y \in U \setminus \{x\}$, we have $\|\partial/\partial\varrho(y)\|_g = 1$, because (with $w := \exp_x^{-1}(y)$ and $\gamma := \operatorname{geod}_{x,w}$)

$$\left\| \dot\gamma(1) \right\|_g = \left\| \dot\gamma(0) \right\|_g = \|w\|_g = \varrho(y);$$

in the first equation, we used that . Moreover, $\partial/\partial\varrho$ indeed is a vector field (check!).

Proposition 3.2.19 (Gauß lemma)**.** *Let (M, g) be a Riemannian manifold, let $x \in M$, and let $U \subset M$ be a normal neighbourhood centred at x. Then, all geodesic spheres around x that lie in U are orthogonal (with respect to g) to the unit radial vector field on U, centred at x.*

Proof. Let $y \in U \setminus \{x\}$ and let $v \in \mathrm{T}_y\, M$ be tangent to the geodesic sphere around x that goes through y. We need to show that $\langle v, \partial/\partial\varrho(y)\rangle_g = 0$.

Because U is a normal neighbourhood of x, there is a $w \in \mathrm{T}_x\, M$ with $\exp_x(w) = y$ and a $\widetilde{v} \in \mathrm{T}_x\, M$ with $d_w \exp_x(\widetilde{v}) = v$. Let $R := \varrho(y)$; then $y \in \operatorname{sphere}_R(x)$. We now consider the radial geodesic

$$\gamma := \operatorname{geod}_{x,w}$$

from x to y. By construction,

$$\dot\gamma(1) = R \cdot \frac{\partial}{\partial\varrho}(y).$$

Thus, in order to prove the Gauß lemma, it remains to show that $\left\langle v, \dot\gamma(1) \right\rangle_g = 0$. We will achieve this through a suitable variation of γ:

As v is tangent to $\mathrm{sphere}_R(x)$, there exists a smooth curve $\eta\colon (-\varepsilon,\varepsilon) \longrightarrow \mathrm{T}_x M$ in the sphere $\partial B_R^{\|\cdot\|_g}(0) \subset \mathrm{T}_x M$ with

$$\eta(0) = w \quad \text{and} \quad \dot\eta(0) = v.$$

We consider the smooth variation

$$\begin{aligned} G\colon (-\varepsilon,\varepsilon) \times I &\longrightarrow M \\ (s,t) &\longmapsto \exp_x\bigl(t \cdot \eta(s)\bigr) = \mathrm{geod}_{x,\eta(s)}(t) \end{aligned}$$

of $\gamma\colon I \longrightarrow M$ and abbreviate $S := \partial_{①} G$ and $T := \partial_{②} G$. By construction,

$$\begin{aligned} \bigl\langle S(0,0), T(0,0)\bigr\rangle_g &= \bigl\langle \partial_{①}G(0,0), \partial_{②}G(0,0)\bigr\rangle_g = \langle 0, \dots\rangle_g = 0, \\ \bigl\langle S(0,1), T(0,1)\bigr\rangle_g &= \bigl\langle \partial_{①}G(0,1), \partial_{②}G(0,1)\bigr\rangle_g = \bigl\langle v, \dot\gamma(1)\bigr\rangle_g \end{aligned}$$

Therefore, it suffices to show that $\langle S(0,\,\cdot\,), T(0,\,\cdot\,)\rangle_g$ is constant; or, equivalently, that $\langle S(0,\,\cdot\,), T(0,\,\cdot\,)\rangle_g' = 0$. For all $t \in I^\circ$, we have

$$\begin{aligned} \bigl\langle S(0,\,\cdot\,), T(0,\,\cdot\,)\bigr\rangle_g'(t) &= \bigl\langle D_\gamma S(0,t), T(0,t)\bigr\rangle_g + \bigl\langle S(0,t), D_\gamma T(0,t)\bigr\rangle_g && \text{(compatibility; Proposition 2.3.7)} \\ &= \bigl\langle D_\gamma S(0,t), T(0,t)\bigr\rangle_g + 0 && (\gamma \text{ is a geodesic}) \\ &= \bigl\langle D_{G(\,\cdot\,,t)} T(0,t), T(0,t)\bigr\rangle_g && \text{(symmetry lemma; Proposition 3.2.4)} \\ &= \frac{1}{2} \cdot \bigl\langle T(0,\,\cdot\,), T(0,\,\cdot\,)\bigr\rangle_g'(t) && \text{(compatibility; Proposition 2.3.7)} \\ &= \frac{1}{2} \cdot \|\dot\gamma\|_g'(t) && \text{(definition of } T \text{ and } G) \\ &= 0, && \text{(Proposition 3.2.14)} \end{aligned}$$

as desired. □

Study note. How many other Gauß lemmas do you know?

Corollary 3.2.20 (gradient of the radial distance)**.** *Let (M,g) be a Riemannian manifold, let $x \in M$, and let $U \subset M$ be a normal neighbourhood centred at x. Then, on $U \setminus \{x\}$, we have*

$$\mathrm{grad}\,\varrho = \frac{\partial}{\partial\varrho}.$$

Proof. Let $y \in U \setminus \{x\}$ and let $v \in \mathrm{T}_y M$. By definition of $\mathrm{grad}\,\varrho$, we need to show that $\mathrm{d}_y\,\varrho(v) = \langle \partial/\partial\varrho(y), v\rangle_g$. Using the Gauß lemma (Proposition 3.2.19), we can decompose

$$v = \lambda \cdot \frac{\partial}{\partial\varrho}(y) + v^\circ$$

where $\lambda \in \mathbb{R}$ and v° is tangential to the geodesic sphere through y, centred at x. We then obtain

$$
\begin{aligned}
\Big\langle \frac{\partial}{\partial\varrho}(y), v\Big\rangle &= \lambda \cdot \Big\langle \frac{\partial}{\partial\varrho}(y), \frac{\partial}{\partial\varrho}(y)\Big\rangle + \Big\langle \frac{\partial}{\partial\varrho}(y), v^\circ\Big\rangle \\
&= \lambda \cdot 1 + 0 && \text{(Remark 3.2.18)} \\
&= \lambda \cdot \mathrm{d}_y\, \varrho\Big(\frac{\partial}{\partial\varrho}(y)\Big) + \mathrm{d}_y\, \varrho(v^\circ) && \text{(Exercise; } v^\circ \text{ is tangent to a level set of } \varrho) \\
&= \mathrm{d}_y\, \varrho(v).
\end{aligned}
$$

Therefore, $\operatorname{grad} \varrho = \partial/\partial\varrho$. □

Corollary 3.2.21 (uniqueness of radial geodesics as minimising curves)**.** *Let (M, g) be a Riemannian manifold, let $x \in M$, let $U \subset M$ be a normal neighbourhood open geodesic ball centred at x, and let $y \in U \setminus \{x\}$. Then, up to affine reparametrisation, the radial geodesic from x to y is the unique minimising curve from x to y.*

Proof. Let $R := \varrho(y)$ and let $\gamma\colon [0, R] \longrightarrow U$ denote the unit speed radial geodesic from x to y.

Moreover, let $\eta\colon [0, S] \longrightarrow M$ be a unit speed curve from x to y. We now show that $L_g(\eta) \geq L_g(\gamma)$ and we characterise the case of equality.

Without loss of generality, we may assume that $\eta(t) \neq x$ for all $t \in (0, S]$. Moreover, we let

$$S_0 := \inf\bigl\{t \in [0, S] \bigm| \eta(t) \in \operatorname{sphere}_R(x)\bigr\}$$

denote the first time that η meets $\operatorname{sphere}_R(x)$. The set on the right-hand side is non-empty because and, by continuity, we have $\eta(S_0) \in \operatorname{sphere}_R(x)$. Using the Gauß lemma (Proposition 3.2.19), we can write

$$\dot\eta = \lambda \cdot \frac{\partial}{\partial\varrho} \circ \eta + V,$$

where $\lambda \in C^\infty([0, S], \mathbb{R})$ and where V is a vector field along η that is tangential to the corresponding geodesic spheres. Then, for all $t \in [0, S]$, we obtain

$$\bigl\|\dot\eta(t)\bigr\|_g^2 \geq \lambda(t)^2 = \mathrm{d}_{\eta(t)}\, \varrho\bigl(\dot\eta(t)\bigr),$$

because $\partial/\partial\varrho$ is a unit vector field, the decomposition is orthogonal, and because $\partial/\partial\varrho = \operatorname{grad} \varrho$ (Corollary 3.2.20). Thus,

$$
\begin{aligned}
L_g(\eta) \geq L_g(\eta|_{[0,S_0]}) &= \lim_{\delta \to 0} \int_\delta^{S_0} \bigl\|\dot\eta(t)\bigr\|_2\, dt \\
&\geq \lim_{\delta \to 0} \int_\delta^{S_0} \mathrm{d}_{\eta(t)}\, \varrho\bigl(\dot\eta(t)\bigr)\, dt = \lim_{\delta\to 0} \int_\delta^{S_0} (\varrho \circ \eta)'(t)\, dt \\
&= \lim_{\delta\to 0} \bigl(\varrho(\eta(S_0)) - \varrho(\eta(\delta))\bigr) && \text{(fundamental theorem of calculus)} \\
&= R - 0 && \text{(choice of } S_0) \\
&= L_g(\gamma). && (\gamma \text{ has unit speed})
\end{aligned}
$$

In particular, γ is a minimising curve from x to y.

Finally, we deal with uniqueness: If $L_g(\eta) = L_g(\gamma)$, then both inequalities in the chain above are equalities. In particular, $S_0 = S$ and $v = 0$. Therefore, $\dot\eta$ is a positive rescaling of $\partial/\partial\varrho \circ \eta$; because η has unit speed, we conclude that

$$\dot\eta = \frac{\partial}{\partial\varrho} \circ \eta.$$

On the other hand, also the radial geodesic γ satisfies $\dot\gamma = \partial/\partial\varrho \circ \gamma$. Because $\eta(0) = \gamma(0)$, the uniqueness of integral curves of vector fields shows $\eta = \gamma$. □

From Corollary 3.2.21 it is not hard to conclude that the radial distance function coincides with the Riemannian distance function and that geodesic spheres and balls in normal neighbourhoods coincide with the corresponding spheres and balls for the Riemannian distance function [18, Corollary 6.12, Corollary 6.13].

It remains to investigate which geodesics are minimising and not only locally minimising. We will do so in Chapter 3.5.3.

Quick check 3.2.22. Are all geodesics minimising curves?
Yes No

3.2.5 Riemannian isometries

Using the relation between Riemannian geodesics and minimising curves, one can also prove the corresponding statement about isomorphisms: Riemannian isometries are the same as metric isometries. This is the explanation why Riemannian geometry can be used to solve "classical" geometric problems, including problems in cartography.

Theorem 3.2.23. *Let (M, g) be a Riemannian manifold. Then the Riemannian isometry group* $\mathrm{Isom}(M, g)$ *coincides with the group of all d_g-isometries.*

Sketch proof. We already know that every Riemannian isometry of (M, g) is a d_g-isometry (Proposition 1.5.8).

Conversely, let $\varphi\colon M \longrightarrow M$ be a bijection that is isometric with respect to d_g. One can then argue as follows [15, Theorem I.11.1]: Using the interplay between geodesics and minimising curves, one can show that φ maps geodesics to geodesics; moreover, this allows to express φ in terms of the exponential map, which gives smoothness. □

Corollary 3.2.24 (the cartography problem)**.** *Round 2-spheres (with the spherical metric) are* not *locally isometric to the Euclidean plane (with the Euclidean metric).*

Proof. Any such local isometry would also be a Riemannian local isometry (Theorem 3.2.23). However, round 2-spheres and the Euclidean plane are *not* locally isometric in the Riemannian sense (because of curvature! Example 2.5.15). □

In particular, in cartography, every flat map of parts of the Earth is a compromise and different types of maps have different advantages and disadvantages.

Study note. Read up on different types of projections in cartography and figure out what their (differential) geometric properties are!

3.3 Completeness

On Riemannian manifolds, we have two canonical notions of completeness: Metric completeness (with respect to the metric induced by the Riemannian metric) and geodesic completeness (i.e., the extendability of geodesics to the whole real line $\mathbb{R}$).

By the Hopf-Rinow theorem, these two notions of completeness coincide. In the following, we will prove the Hopf-Rinow theorem and some important consequences.

3.3.1 Two notions of completeness

Definition 3.3.1 (metric completeness, geodesic completeness)**.** Let (M, g) be a Riemannian manifold.

- The Riemannian manifold (M, g) is *metrically complete*, if M is connected and complete as a metric space with respect to the induced metric d_g (Theorem 1.5.13).
- The Riemannian manifold (M, g) is *geodesically complete*, if for all $x \in M$ and all $v \in \mathrm{T}_x M$, the maximal geodesic $\mathrm{geod}_{x,v}^{(M,g)}$ is defined on all of $\mathbb{R}$.

Quick check 3.3.2. Let $n \in \mathbb{N}_{\geq 1}$. The following subsets of $\mathbb{R}^n$ are equipped with the Euclidean Riemannian metric.

1. Is $\mathbb{R}^n$ metrically complete?

   Yes No

2. Is $\mathbb{R}^n$ geodesically complete?

   Yes No

3. Is $\mathbb{R}^n \setminus \{0\}$ metrically complete?

   Yes No

4. Is $\mathbb{R}^n \setminus \{0\}$ geodesically complete?

   Yes No

Example 3.3.3. All compact connected Riemannian manifolds are metrically complete (check!).

3.3.2 The Hopf-Rinow theorem

Theorem 3.3.4 (Hopf-Rinow). *Let (M, g) be a non-empty connected Riemannian manifold. Then the following are equivalent:*

1. *The Riemannian manifold (M, g) is geodesically complete.*
2. *For all $x \in M$, we have $\operatorname{Exp}_x = \operatorname{T}_x M$.*
3. *There exists an $x \in M$ with $\operatorname{Exp}_x = \operatorname{T}_x M$.*
4. *The Riemannian manifold (M, g) is metrically complete.*

The key step in the proof of the Hopf-Rinow theorem is:

Proposition 3.3.5. *Let (M, g) be a connected Riemannian manifold and let $x \in M$ with $\operatorname{Exp}_x = \operatorname{T}_x M$. Then, for every $y \in M$, there exists a minimising geodesic from x to y.*

Proof. Let $y \in M$. A geodesic $\gamma\colon [a, b] \longrightarrow M$ with $\gamma(a) = x$ *aims at* y if γ is minimising and

$$d_g(x, y) = d_g\bigl(x, \gamma(b)\bigr) + d_g\bigl(\gamma(b), y\bigr).$$

In the following, we will show that there exists a geodesic from x that aims at y and has length $d_g(x, y)$. This suffices, because then

$$\begin{aligned} d_g\bigl(\gamma(b), y\bigr) &= d_g(x, y) - d_g\bigl(x, \gamma(b)\bigr) && (\gamma \text{ aims at } y)\\ &= d_g(x, y) - L_g(\gamma) && (\gamma \text{ is minimising})\\ &= d_g(x, y) - d_g(x, y)\\ &= 0, \end{aligned}$$

and so $\gamma(b) = y$ (and γ is minimising, because it aims at y).

We find such a geodesic by starting with a suitable radial geodesic: Let $r \in \mathbb{R}_{>0}$ be so small that there exists a normal neighbourhood closed geodesic ball $\overline{\operatorname{ball}}_r(x)$ around x; without loss of generality, we may assume that $y \notin \overline{\operatorname{ball}}_r(x)$ (otherwise, we can apply Corollary 3.2.21). Continuity of d_g and compactness of the geodesic sphere $\operatorname{sphere}_r(x)$ show that there exists a point $z \in \mathrm{sphere}_r(x)$ that minimises $d_g(\,\cdot\,, y)$ on $\mathrm{sphere}_r(x)$. Let $\gamma\colon I \longrightarrow M$ be the maximal unit speed geodesic such that $\gamma|_{[0,r]}$ coincides with the unit speed radial geodesic from x to z (Figure 3.4).

We then show that

① the initial segment $\gamma|_{[0,r]}$ aims at y and then that

② $\gamma|_{[0,R]}$ aims at y, where $R := d_g(x, y)$.

Ad ①. Because $\gamma|_{[0,r]}$ is a radial geodesic, we know that $\gamma|_{[0,r]}$ is minimising (Corollary 3.2.21). Thus, it remains to establish that

$$d_g(x, y) = d_g(x, z) + d_g(z, x).$$

Clearly, "$\leq$" holds by the triangle inequality. For "$\geq$", let $\eta\colon [a, b] \longrightarrow M$ be a piecewise regular curve from x to y and let $t \in [a, b]$ be the first time with $d_g(x, \eta(t)) = r$ (such a time exists; check!). Then, we have

$$\begin{aligned} L_g(\eta) &= L_g(\eta|_{[a,t]}) + L_g(\eta|_{[t,b]}) \\ &\geq d_g\bigl(x, \eta(t)\bigr) + d_g\bigl(\eta(t), y\bigr) && \text{(by definition of } d_g) \\ &\geq r + d_g(z, y) && \text{(by definition of } z = \eta(t)) \\ &= d_g(x, z) + d_g(z, y). \end{aligned}$$

Taking the infimum over all such curves η, we obtain $d_g(x, y) \geq d_g(x, z) + d_g(z, x)$. Thus, $\gamma|_{[0,r]}$ aims at y.

Ad ②. To complete the proof, we extend this result to show that $\gamma|_{[0,R]}$ aims at y. We use the common strategy of looking at the set

$$J := \bigl\{t \in [0, R] \bigm| \gamma|_{[0,t]} \text{ aims at } y\bigr\}$$

of "good" times and setting $S := \sup J$. By ①, the set J is non-empty: we have $r \in J$.

Assume for a contradiction that $S < R$. Let $x' := \gamma(S)$ and let $r' \in \mathbb{R}_{>0}$ be so small that x' admits a normal neighbourhood closed geodesic ball $\overline{\mathrm{ball}}_{r'}(x')$. Again, let $z' \in \mathrm{sphere}_{r'}(x')$ be a point that minimises $d(\,\cdot\,, y)$ on $\mathrm{sphere}_{r'}(x')$ and let $\eta\colon [0, r'] \longrightarrow M$ be the radial geodesic from x' to z'. By the argument of ①, η aims from x' at y.

The concatenation of $\gamma|_{[0,S]}$ and η is a minimising curve (check the distances of the endpoints!), whence a geodesic (Corollary 3.2.12). Then, by uniqueness of geodesics, this concatenation must be $\gamma|_{[0,S+r']}$. Moreover, we have

$$\begin{aligned} d_g\bigl(\gamma(0), y\bigr) &= d_g\bigl(\gamma(0), \gamma(S)\bigr) + d_g\bigl(\gamma(S), y\bigr) && (\gamma|_{[0,S]} \text{ aims at } y) \\ &= d_g\bigl(\gamma(0), \gamma(S)\bigr) + d_g\bigl(\gamma(S), \gamma(S + r')\bigr) + d_g\bigl(\gamma(S + r'), y\bigr) && (\eta \text{ aims from } x' \text{ at } y) \\ &= d_g\bigl(\gamma(0), \gamma(S + r')\bigr) + d_g\bigl(\gamma(S + r'), y\bigr), && (\gamma|_{[0,S+r']} \text{ is minimising}) \end{aligned}$$

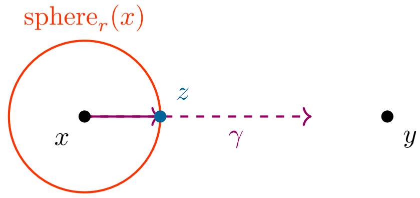


Figure 3.4.: The proof of Proposition 3.3.5, schematically

and so $\gamma|_{[0,S+r']}$ aims at y, contradicting the maximality of S. This contradiction shows that $S = R$. $\square$

Proof of the Hopf-Rinow theorem (Theorem 3.3.4). Ad 1. $\Longrightarrow$ 2. This is immediate from the definition of geodesic completeness and the exponential map.

Ad 2. $\Longrightarrow$ 3. This is clear.

Ad 3. $\Longrightarrow$ 4. Let us suppose that there exists an $x \in M$ with $\operatorname{Exp}_x = \mathrm{T}_x M$. We show that M is metrically complete with respect to d_g: Let $(y_n)_{n\in\mathbb{N}}$ be a d_g-Cauchy sequence in M. In view of Proposition 3.3.5, for each $n \in \mathbb{N}$, there exists a unit speed minimising geodesic $\gamma_n \colon [0, D_n] \longrightarrow M$ with

$$D_n = d_g(x, y_n) \quad \text{and} \quad \gamma_n(0) = x \quad \text{and} \quad \gamma_n(D_n) = y_n.$$

Then γ_n is of the form

$$\gamma_n = \operatorname{geod}_{x,v_n}|_{[0,D_n]}$$

for a suitable g-unit vector $v_n \in \mathrm{T}_x M$. Because $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, the sequence $(D_n)_{n\in\mathbb{N}}$ is bounded (check!), and so the sequence $(D_n \cdot v_n)_{n\in\mathbb{N}}$ in $\mathrm{T}_x M$ is bounded with respect to g. Therefore, this sequence has an accumulation point v in $\mathrm{T}_x M$.

We now show that $y := \exp_x(v)$ is an accumulation point of $(y_n)_{n\in\mathbb{N}}$:

On the one hand, continuity of $\exp_x$ yields that y is an accumulation point of $(\exp_x(D_n \cdot v_n))_{n\in\mathbb{N}}$. On the other hand, by construction, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \exp_x(D_n \cdot v_n) &= \operatorname{geod}_{x,D_n\cdot v_n}(1) && \text{(definition of } \exp_x\text{)} \\ &= \operatorname{geod}_{x,v_n}(D_n) && \text{(Theorem 3.1.5)} \\ &= y_n. && \text{(choice of } v_n \text{ and } D_n\text{)} \end{aligned}$$

Thus, y is an accumulation point of $(y_n)_{n\in\mathbb{N}}$. As $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, it follows that y is the d_g-limit of $(y_n)_{n\in\mathbb{N}}$. In particular, (M, d_g) is complete.

Ad 4. $\Longrightarrow$ 1. Let (M, g) be metrically complete. *Assume* for a contradiction that (M, g) is *not* geodesically complete. Thus, there exists a unit speed geodesic $\gamma \colon I \longrightarrow M$ that does not admit an extension beyond $b := \sup I < \infty$. We choose an increasing sequence $(t_n)_{n\in\mathbb{N}}$ in I° with $\lim_{n\to\infty} t_n = b$ and set $y_n := \gamma(t_n)$ for each $n \in \mathbb{N}$.

- Then, $(y_n)_{n\in\mathbb{N}}$ is a d_g-Cauchy sequence in M, because for all $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned} d_g(y_n, y_m) &\leq L_g(\gamma|_{[t_n,t_m]}) && \text{(definition of } d_g\\ &= |t_n - t_m|. && (\gamma \text{ has unit speed)} \end{aligned}$$

  As M is complete with respect to d_g, there exists a d_g-limit $y \in M$ of $(y_n)_{n\in\mathbb{N}}$.

- Let $U \subset M$ be a uniformly normal neighbourhood of y (Proposition 3.1.14), say for radius ε. Because $\lim_{n\to\infty} y_n = y$, there exists an $n \in \mathbb{N}$ with $t_n > b - \varepsilon$, $t_n - \varepsilon \in I^\circ$, and $y_n \in U$.

Finally, we extend γ by a suitable radial geodesic: Let

$$\begin{aligned} \widetilde{\gamma} \colon I \cup (t_n - \varepsilon, t_n + \varepsilon) &\longrightarrow M \\ t &\longmapsto \begin{cases} \gamma(t) & \text{if } t \in I^\circ \\ \operatorname{geod}_{y_n,\dot{\gamma}(t-t_n)}(t - t_n) & \text{if } t \in (t_n - \varepsilon, t_n + \varepsilon). \end{cases} \end{aligned}$$

On the intersection, both branches give the same values (by the uniqueness of geodesics; Theorem 2.2.22). Thus, $\widetilde{\gamma}$ is a geodesic that strictly extends γ "to the right", which contradicts our assumption. Hence, (M, g) is geodesically complete. □

Quick check 3.3.6. In the proof of Theorem 3.3.4, did we use the hypothesis that M is non-empty at all?
Yes No

Study note. Try to find the original formulation of the Hopf-Rinow theorem in the literature.

Corollary 3.3.7. *Let (M, g) be a connected Riemannian manifold.*

1. *If (M, g) is metrically complete, then each two points in M can be joined by a minimising geodesic.*
2. *If M is compact, then each two points in M can be joined by a minimising geodesic.*

Proof. The second part follows from the first part, because all compact Riemannian manifolds are metrically complete (Example 3.3.3).

For the first part: If (M, g) is metrically complete, and $x, y \in M$, then $\operatorname{Exp}_x = \mathrm{T}_x M$, by the Hopf-Rinow theorem (Theorem 3.3.4). Therefore, there exists a minimising geodesic from x to y (Proposition 3.3.5). □

3.4 Model spaces

We quickly determine the geodesics and isometry groups of the model spaces. In particular, we will also put this into the context of the parallel postulate.

3.4.1 Geodesics of the model spaces

The geodesics of the model spaces can be computed directly through the geodesic equation and explicit formulae for the Christoffel symbols. Alternatively, if we are only interested in the images of geodesics (and not in specific parametrisations), we can also exploit symmetry and the following observation:

Proposition 3.4.1 (fixed sets are geodesic). *Let (M, g) be a Riemannian manifold and let $N \subset M$ be a connected one-dimensional smooth submanifold for which there exists a $\varphi \in \operatorname{Isom}(M, g)$ with*

$$N = \{x \in M \mid \varphi(x) = x\}.$$

Moreover, let $x \in N$. Then, for each of the two unit vectors $v \in \mathrm{T}_x N \subset \mathrm{T}_x M$, we have

$$\operatorname{im}(\operatorname{geod}_{x,v}) = N.$$

Proof. As a first step, we can use uniqueness of geodesics to derive that $\varphi \circ \operatorname{geod}_{x,v} = \operatorname{geod}_{x,v}$. The hypothesis on N and a connectedness/extension argument then shows the claim (Exercise). □

Example 3.4.2 (geodesics of Euclidean spaces). Let $n \in \mathbb{N}_{>0}$. Then the maximal geodesics of $\mathbb{R}^n$ (with the Euclidean Riemannian metric) are exactly the affine lines (Example 2.2.25).

Example 3.4.3 (geodesics of round spheres). Let $n \in \mathbb{N}_{>0}$. The images of maximal geodesics of the round sphere $\mathbb{S}^n$ are exactly the *great circles* of $\mathbb{S}^n$, i.e., the intersections of $\mathbb{S}^n$ with planes in $\mathbb{R}^{n+1}$ (going through the origin):

- Every great circle is the image of a maximal geodesic of $\mathbb{S}^n$: We can apply Proposition 3.4.1 to the orthogonal reflection at the corresponding plane (check!). Alternatively, we can also use Example 2.2.25.
- There are no other images of maximal geodesics: Each maximal geodesic is uniquely determined by an initial value and and initial direction; the great circles through a given point already sweep out all these possibilities (check!).

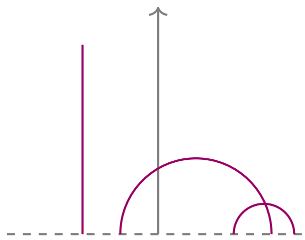


Figure 3.5.: Maximal geodesics in the Poincaré halfspace model of hyperbolic space

Example 3.4.4 (geodesics of hyperbolic spaces)**.** Let $n \in \mathbb{N}_{>0}$. The images of maximal geodesics of the Poincaré disk model $\mathbb{B}^n(1)$ that go through 0 are exactly the lines in $\mathbb{B}^n(1)$ through 0:

- Every such line is the image of a maximal geodesic of $\mathbb{B}^n(1)$: We can apply Proposition 3.4.1 to the orthogonal reflection at this line (check!).
- There are no other images of maximal geodesics through 0: Each maximal geodesic through 0 is uniquely by its parameter and direction at 0; the lines in $\mathbb{B}^n(1)$ through 0 already sweep out all these possibilities (check!).

Using the Cayley transform and horizontal translations as well as dilations, one can conclude (after some lengthy, but elementary, computations) that the images of maximal geodesics of the Poincaré halfspace model $\mathbb{U}^n(1)$ are exactly

- the vertical lines in $\mathbb{U}^n(1)$ and
- the semi-circles in $\mathbb{U}^n(1)$ whose centre lies on a point in $\mathbb{R}^{n-1} \times \{0\}$

(Figure 3.5). In the Poincaré disk model, the images of maximal geodesics are exactly the lines through the centre and the circle arcs that meet the boundary sphere orthogonally.

In particular, from this, one can also derive that $\mathbb{H}^n$ is geodesically complete (check!), whence complete (by the Hopf-Rinow theorem; Theorem 3.3.4).

In principle, from these considerations, we can also figure out parametrisations of the geodesics on the model spaces. These parametrisations then give explicit formulae for the computation of the Riemannian distance function on the model spaces. However, in practice, it is often better to avoid these explicit descriptions of the Riemannian distance function and to try to use other arguments.

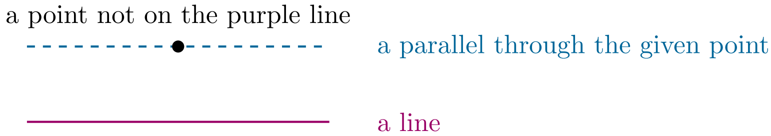


Figure 3.6.: The parallel postulate, schematically

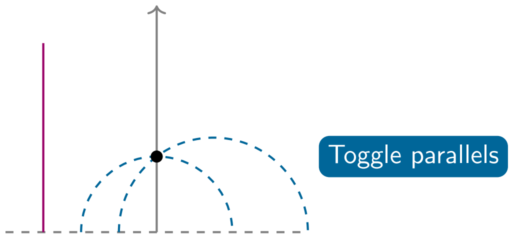


Figure 3.7.: The parallel postulate does *not* hold in $\mathbb{H}^2$

Remark 3.4.5 (the parallel postulate)**.** In the context of Riemannian geometry, lines are modelled by geodesic lines (defined on all of $\mathbb{R}$). Thus, in this language, the *parallel postulate* reads (Figure 3.6):

> A Riemannian manifold (M, g) satisfies the *parallel postulate*, if the following holds: For every geodesic line γ and every $x \in M \setminus \operatorname{im} \gamma$, there exists a unique geodesic line η with
>
> $$x \in \operatorname{im} \eta \quad \text{and} \quad \operatorname{im} \eta \cap \operatorname{im} \gamma = \emptyset.$$

In axiomatic geometry, it used to be a long-standing open problem to determine whether the parallel postulate follows from the other axioms or whether there exist geometries that satisfy all the other axioms but do *not* satisfy the parallel postulate.

In modern geometry, this question is not hard to answer, because we have a rich collection of tools to construct various geometries. For example, the hyperbolic plane $\mathbb{H}^2$ satisfies all of (Hilbert's exact version of) Euclid's axioms except for the parallel postulate. That the parallel postulate fails to hold in $\mathbb{H}^2$ can be seen from the computation of geodesics in Example 3.4.4 and the constellation in Figure 3.7.

3.4.2 Isometries of the model spaces

For the sake of completeness, we round this discussion off with a description of the isometry groups of the model spaces. Because the Riemannian isometry group coincides with the metric isometry group (Theorem 3.2.23), these results also compute the isometry groups of the model spaces in the metric sense.

Theorem 3.4.6. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} \varphi\colon \mathbb{R}^n \rtimes \mathrm{O}(n) &\longrightarrow \mathrm{Isom}(\mathbb{R}^n, \textit{Euclidean Riemannian metric}) \\ (v, A) &\longmapsto (x \mapsto A \cdot x + v) \end{aligned}$$

is an isomorphism.

Proof. We already know that this map is an injective group homomorphism (Chapter 1.4.2). Why do we have surjectivity?

Let $f \in \mathrm{Isom}(\mathbb{R}^n)$; then, clearly, the matrix A that represents $\mathrm{d}_0 f$ with respect to the standard basis lies in $\mathrm{O}(n)$. Because isometries are determined by the initial data at a single point (Proposition 3.1.15), we obtain that

$$f = \varphi\bigl(f(0), A\bigr). \qquad \square$$

Theorem 3.4.7. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} \varphi\colon\ \mathrm{O}(n+1) &\longrightarrow \mathrm{Isom}(\mathbb{S}^n, \textit{round metric of radius } 1) \\ A &\longmapsto (x \mapsto A \cdot x) \end{aligned}$$

is an isomorphism.

Proof. The map φ is an injective group homomorphism (Proposition 1.4.10). Similarly to the proof in the Euclidean case (Theorem 3.4.6), the arguments in the proof of Proposition 1.4.10 show in combination with Proposition 3.1.15 that φ is surjective (check!). $\square$

The same methods also show that the isometry group of $\mathbb{H}^n$ is isomorphic to $\mathrm{SO}(n, 1)$.

3.5 Jacobi fields

We now take the investigation of geodesics to the next level.

- We will study the second derivative of the length functional on variations of geodesics; this leads to the second variation formula and will eventually explain on which intervals geodesics are minimising.

- We will study variations of geodesics through geodesics. The corresponding vector fields driving the evolution of such variations are called Jacobi fields; Jacobi fields will be our key tool in proving local-global results in Chapter 4.

3.5.1 The second variation formula and the index form

Every minimising curve is a critical point of the length functional (Theorem 3.2.11). For a refined analysis of the relation between critical points and local minima, we look at the second derivative of the length functional on variations of geodesics.

The first variation formula (Theorem 3.2.8) describes how the variation field controls the evolution of length along variations. Similarly, the second variation formula describes how the normal part of the variation field controls the evolution of the change of lengths along variations of geodesics. As always when second derivatives come into play, we have to expect a contribution of

The integrand of the second variation formula leads to a bilinear form, the index form. This index form gives practical criteria for geodesics to be (non-)minimising.

We start by quickly introducing the normal component of vector fields along curves.

Remark 3.5.1 (normal component). Let (M, g) be a Riemannian manifold, let $\gamma\colon [a, b] \longrightarrow M$ be a unit speed geodesic, and let $V \in \Gamma(\mathrm{T}\, M|\gamma)$. Then

$$\overline{V} := \langle V, \dot{\gamma}\rangle_g \cdot \dot{\gamma}$$

is a vector field along γ, the *tangential component* of V. Consequently,

$$V^\perp := V - \overline{V}$$

is a vector field along γ that is orthogonal to the velocity field $\dot{\gamma}$. The vector field $V^\perp$ is the *normal component* of V. We call a vector field along a curve *normal* if its tangential component is zero.

Straightforward calculations show that (Exercise)

$$\begin{aligned} D_\gamma \overline{V} &= \overline{D_\gamma V} \\ D_\gamma(V^\perp) &= (D_\gamma V)^\perp \\ \|D_\gamma V\|_g^2 &= \langle D_\gamma V, \dot{\gamma}\rangle_g^2 + \big\|(D_\gamma V)^\perp\big\|_g^2 \\ \mathrm{Rm}(V, \dot{\gamma}, \dot{\gamma}, V) &= \mathrm{Rm}(V^\perp, \dot{\gamma}, \dot{\gamma}, V^\perp). \end{aligned}$$

Theorem 3.5.2 (the second variation formula). *Let (M, g) be a Riemannian manifold, let $\gamma\colon [a, b] \longrightarrow M$ be a unit speed geodesic on M, let G be a proper* *variation of γ, and let V be the variation field of G. Then*

$$\frac{\partial^2}{\partial s^2}\Big|_{s=0} L_g\big(G(s,\,\cdot\,)\big) = \int_a^b \big\|D_\gamma(V^\perp)(t)\big\|_g^2 - \mathrm{Rm}\big(V^\perp(t),\dot\gamma(t),\dot\gamma(t),V^\perp(t)\big)\,dt.$$

Study note. Recall the proof of the first variation formula (Theorem 3.2.8). In addition to basic properties of the various types of differentiation, we also used another ingredient:

As the second variation formula involves second derivatives, we need a corresponding symmetry property for iterated covariant derivatives. As expected, this contains a curvature contribution.

Proposition 3.5.3 (symmetry lemma II: iterated covariant derivatives). *Let (M,g) be a Riemannian manifold, let $G\colon(-\varepsilon,\varepsilon)\times[a,b]\longrightarrow M$ be a smooth regular family of curves on M, let V be a smooth vector field along G, and let $S := \partial_{①}G$ and $T := \partial_{②}G$. Then, for all $(s,t)\in(-\varepsilon,\varepsilon)\times[a,b]$, we have*

$$D_{G(\,\cdot\,,t)}D_{G(s,\,\cdot\,)}V(s,t) - D_{G(s,\,\cdot\,)}D_{G(\,\cdot\,,t)}V(s,t) = \mathrm{R}\big(S(s,t),T(s,t),V(s,t)\big).$$

Proof. As in the proof of the symmetry lemma (Proposition 3.2.4), we prove this through a computation in local coordinates on some normal neighbourhood. Without loss of generality, we may thus assume that M is a normal neighbourhood around a point x in M. Let $(E_1,\dots,E_n)$ be the coordinate frame associated with normal coordinates, centred at x. Then we can write V in the form

$$V = \sum_{j=1}^n V^j\cdot E_j\circ G.$$

Thus, for all (s,t), we have

$$D_{G(s,\,\cdot\,)}V(s,t) = \sum_{j=1}^n\big(\partial_2 V^j(s,t)\cdot E_j\circ G(s,t) + V^j(s,t)\cdot D_{G(s,\,\cdot\,)}E_j(G(s,t))\big)$$

and so

$$\begin{aligned} D_{G(\,\cdot\,,t)}D_{G(s,\,\cdot\,)}V(s,t) = \sum_{j=1}^n\big(&\partial_1\partial_2 V^j(s,t)\cdot E_j\circ G(s,t)\\ &+ \partial_2 V^j(s,t)\cdot D_{G(\,\cdot\,,t)}E_j(G(s,t))\\ &+ \partial_1 V^j(s,t)\cdot D_{G(s,\,\cdot\,)}E_j(G(s,t))\\ &+ V^j(s,t)\cdot D_{G(\,\cdot\,,t)}D_{G(s,\,\cdot\,)}E_j(G(s,t))\big). \end{aligned}$$

Therefore, we obtain for our main difference (Schwarz!):

$$
\begin{aligned}
&D_{G(\,\cdot\,,t)}D_{G(s,\,\cdot\,)}V(s,t) - D_{G(s,\,\cdot\,)}D_{G(t,\,\cdot\,)}V(s,t)\\
&= \sum_{j=1}^{n} V^j(s,t)\cdot\big(D_{G(\,\cdot\,,t)}D_{G(s,\,\cdot\,)}E_j(G(s,t)) - D_{G(s,\,\cdot\,)}D_{G(\,\cdot\,,t)}E_j(G(s,t))\big)
\end{aligned}
$$

For $j \in \{1,\ldots,n\}$, let

$$x^j := \varphi^j \circ G$$

denote the j-th coordinate of G with respect to φ. In this notation,

$$S = \boxed{?}$$

$$T = \boxed{?}$$

As E_j and $\nabla_{E_k}E_j$ are extendable vector fields, we obtain

$$D_{G(s,\,\cdot\,)}E_j(G(s,t)) = \sum_{k=1}^{n} \partial_2 x^k(s,t)\cdot \nabla_{E_k}E_j(G(s,t))$$

and

$$
\begin{aligned}
&D_{G(\,\cdot\,,t)}D_{G(s,\,\cdot\,)}E_j(G(s,t))\\
&\quad= \sum_{k=1}^{n} D_{G(\,\cdot\,,t)}\big(\partial_2 x^k(s,t)\cdot\nabla_{E_k}E_j\big)(G(s,t))\\
&\quad= \sum_{k=1}^{n}\Bigg(\partial_1\partial_2 x^k(s,t)\cdot\nabla_{E_k}E_j(G(s,t)) + \sum_{\ell=1}^{n}\partial_2 x^k(s,t)\cdot\partial_1 x^\ell(s,t)\cdot\nabla_{E_\ell}\nabla_{E_k}E_j(G(s,t))\Bigg).
\end{aligned}
$$

Together with the Schwarz theorem, we conclude that

$$
\begin{aligned}
&D_{G(\,\cdot\,,t)}D_{G(s,\,\cdot\,)}E_j(G(s,t)) - D_{G(s,\,\cdot\,)}D_{G(\,\cdot\,,t)}E_j(G(s,t))\\
&\quad= \sum_{k=1}^{n}\sum_{\ell=1}^{n}\partial_2 x^k(s,t)\cdot\partial_1 x^\ell(s,t)\cdot\big(\nabla_{E_\ell}\nabla_{E_k}E_j(G(s,t)) - \nabla_{E_k}\nabla_{E_\ell}E_j(G(s,t))\big)\\
&\quad= \sum_{k=1}^{n}\sum_{\ell=1}^{n}\partial_2 x^k(s,t)\cdot\partial_1 x^\ell(s,t)\cdot\mathrm{R}\big(E_\ell\circ G(s,t), E_k\circ G(s,t), E_j\circ G(s,t)\big) \qquad ((E_j)_j \text{ is a coordinate frame})\\
&\quad= \mathrm{R}\big(S(s,t), T(s,t), E_j\circ G(s,t)\big).
\end{aligned}
$$

Summing up all these contributions hence gives the desired symmetry. □

Proof of Theorem 3.5.2. We proceed as in the proof of the first variation formula (Theorem 3.2.8): We write $S := \partial_{\textcircled{1}}G$ and $T := \partial_{\textcircled{2}}G$. Let $[\widetilde{a},\widetilde{b}] \subset [a,b]$ be an interval on which G is smooth. The proof of the first variation formula

(Theorem 3.2.8) shows that

$$\frac{\partial}{\partial s}\Big|_{s=\sigma} L_g\big(G(s,\ \cdot\ )|_{[\widetilde{a},\widetilde{b}]}\big) = \int_{\widetilde{a}}^{\widetilde{b}} \frac{\langle D_{G(\sigma,\ \cdot\ )}S(\sigma,t), T(\sigma,t)\rangle_g}{\langle T(\sigma,t), T(\sigma,t)\rangle_g^{1/2}}\, dt$$

for all $\sigma \in (-\varepsilon,\varepsilon)$. Differentiating once more, we therefore obtain

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\Big|_{s=0} L_g\big(G(s,\ \cdot\ )|_{[\widetilde{a},\widetilde{b}]}\big) &= \int_{\widetilde{a}}^{\widetilde{b}} \frac{\partial_1 u(0,t)\cdot v(0,t) - u(0,t)\cdot \partial_1 v(0,t)}{v(0,t)^2}\, dt && \text{(elementary analysis)}\\ &= \int_{\widetilde{a}}^{\widetilde{b}} \partial_1 u(0,t) - u(0,t)\cdot \partial_1 v(0,t)\, dt && (\gamma \text{ has unit speed})\end{aligned}$$

where

$$\begin{aligned} u(\sigma,t) &:= \big\langle D_{G(\sigma,\ \cdot\ )}S(\sigma,t), T(\sigma,t)\big\rangle_g\\ v(\sigma,t) &:= \langle T(\sigma,t), T(\sigma,t)\rangle_g^{1/2}.\end{aligned}$$

The two parts of the above integral can be computed as follows:

- For all $t \in (\widetilde{a},\widetilde{b})$, we have

$$\begin{aligned} u(0,t)\cdot\partial_1 v(0,t) &= \big\langle D_\gamma V(t), \dot\gamma(t)\big\rangle_g \cdot \big\langle D_{G(\ \cdot\ ,t)}T(0,t), \dot\gamma(t)\big\rangle_g && \text{(compatibility and chain rule)}\\ &= \big\langle D_\gamma V(t), \dot\gamma(t)\big\rangle_g \cdot \big\langle D_{G(0,\ \cdot\ )}S(0,t), \dot\gamma(t)\big\rangle_g && \text{(symmetry lemma; Proposition 3.2.4)}\\ &= \big\langle D_\gamma V(t), \dot\gamma(t)\big\rangle_g^2\\ &= \big\|D_\gamma V(t)\big\|_g^2 - \big\|D_\gamma(V^\perp)(t)\big\|_g^2. && (\gamma \text{ has unit speed; Remark 3.5.1)}\end{aligned}$$

- Moreover, for all $t \in (\widetilde{a},\widetilde{b})$, we have

$$\begin{aligned} \partial_1 u(0,t) &= \big\langle D_{G(\ \cdot\ ,t)}D_\gamma S(0,t), \dot\gamma(t)\big\rangle_g + \big\langle D_\gamma V(t), D_{G(\ \cdot\ ,t)}T(0,t)\big\rangle_g && \text{(compatibility for } D_{G(\ \cdot\ ,t)})\\ &= \big\langle D_\gamma D_{G(\ \cdot\ ,t)}S(0,t), \dot\gamma(t)\big\rangle_g + \big\langle D_\gamma V(t), D_\gamma S(0,t)\big\rangle_g\\ &\quad + \big\langle \mathrm{R}(V(t),\dot\gamma(t),V(t)),\dot\gamma(t)\big\rangle_g && \text{(symmetry lemmas; Proposition 3.5.3/3.2.4)}\\ &= \big\langle D_\gamma D_{G(\ \cdot\ ,t)}S(0,t), \dot\gamma(t)\big\rangle_g + \big\|D_\gamma V(t)\big\|_g^2\\ &\quad - \mathrm{Rm}\big(V^\perp(t),\dot\gamma(t),\dot\gamma(t),V^\perp(t)\big). && \text{(Remark 3.5.1)}\end{aligned}$$

Putting the terms back together, we obtain that

$$\frac{\partial^2}{\partial s^2}\Big|_{s=0} L_g\big(G(s,\ \cdot\ )|_{[\widetilde{a},\widetilde{b}]}\big) = \int_{\widetilde{a}}^{\widetilde{b}} \big\|D_\gamma(V^\perp)\big\|_g^2 - \mathrm{Rm}(V^\perp,\dot\gamma,\dot\gamma,V^\perp) + \int_{\widetilde{a}}^{\widetilde{b}} \big\langle D_\gamma D_{G(\ \cdot\ ,t)}S(0,t),\dot\gamma(t)\big\rangle_g\, dt.$$

What can we do with this last integral? Using compatibility and the fact that γ is a geodesic, one recognises that the integrand of this last integral is nothing but the derivative of $t \mapsto \langle D_{G(\,\cdot\,,t)}S(0,t), \dot\gamma(t)\rangle_g$ (check!). Therefore, the fundamental theorem of calculus yields

$$\int_{\widetilde a}^{\widetilde b} \bigl\langle D_\gamma D_{G(\,\cdot\,,t)}S(0,t), \dot\gamma(t)\bigr\rangle_g \,dt = \bigl\langle D_{G(\,\cdot\,,t)}S(0,t), \dot\gamma(t)\bigr\rangle_g\Big|_{t=\widetilde a}^{t=\widetilde b}.$$

The function $t \mapsto D_{G(\,\cdot\,,t)}S(0,t)$ is continuous (check!) and has the value 0 at a and b (check!). Therefore, these contributions over the whole interval $[a,b]$ sum up to zero.

As the length is additive with respect to concatenation of intervals and G is piecewise smooth, the second variation formula is proved. □

Definition 3.5.4 (index form)**.** Let (M,g) be a Riemannian manifold and let $\gamma\colon [a,b] \longrightarrow M$ be a geodesic on M. For proper normal vector fields V, W along γ, we define the *index* by

$$\mathrm{I}_\gamma(V,W) := \int_a^b \bigl(\langle D_\gamma V, D_\gamma W\rangle_g - \mathrm{Rm}(V,\dot\gamma,\dot\gamma,W)\bigr) \in \mathbb{R}.$$

Corollary 3.5.5 (index criterion)**.** *Let (M,g) be a Riemannian manifold and let γ be a unit speed geodesic on M.*

1. *If G is a proper variation of γ with normal variation field V, then*

$$\frac{\partial^2}{\partial s^2}\Big|_{s=0} L_g\bigl(G(s,\,\cdot\,)\bigr) = \mathrm{I}_\gamma(V,V).$$

2. *In particular: If γ is minimising, then every proper normal vector field V along γ satisfies $\mathrm{I}_\gamma(V,V) \geq 0$.*

Proof. Ad 1. This is a direct consequence of the second variation formula (Theorem 3.5.2).

Ad 2. Every proper vector field along γ can be viewed as the variation field of a variation G of γ (Proposition 3.2.7). As γ is minimising and $(s \mapsto L_g(G(s,\,\cdot\,)))$ is smooth, we have $\partial^2/\partial s^2|_{s=0} L_g(G(s,\,\cdot\,)) \geq 0$. Applying the first part finishes the proof. □

Effective use of the index criterion (Corollary 3.5.5) is possible through Jacobi fields. As a preparation, we give a slightly more explicit description of the index form:

Proposition 3.5.6 (index form, expanded version)**.** *Let (M,g) be a Riemannian manifold, let $\gamma\colon [a,b] \longrightarrow M$ be a geodesic on M, and let V, W be proper normal piecewise smooth vector fields along γ. Then*

$$\mathrm{I}_\gamma(V,W) = -\int_a^b \bigl\langle D_\gamma D_\gamma V + \mathrm{R}(V,\dot\gamma,\dot\gamma), W\bigr\rangle_g - \sum_{j=1}^k \bigl\langle \Delta_j D_\gamma V, W(a_j)\bigr\rangle_g,$$

where $\{a_a, \dots, a_k\}$ are the non-smooth points of V and the $\Delta_j D_\gamma V$ are the "jumps" of $D_\gamma V$ at a_j.

Proof. Let $[\widetilde{a}, \widetilde{b}] \subset [a, b]$ be an interval on which both V and W are smooth. Then, compatibility of the connection yields (Proposition 2.3.7)

$$\langle D_\gamma V, W\rangle_g' = \langle D_\gamma D_\gamma V, W\rangle_g + \langle D_\gamma V, D_\gamma W\rangle_g.$$

Therefore, we obtain

$$\begin{aligned}\int_{\widetilde{a}}^{\widetilde{b}} \bigl(\langle D_\gamma V, D_\gamma W\rangle - \mathrm{Rm}(V, \dot\gamma, \dot\gamma, W)\bigr) &= \int_{\widetilde{a}}^{\widetilde{b}} \langle D_\gamma V, W\rangle_g' - \int_{\widetilde{a}}^{\widetilde{b}} \bigl(\langle D_\gamma D_\gamma V, W\rangle + \bigl\langle \mathrm{R}(V, \dot\gamma, \dot\gamma), W\bigr\rangle_g\bigr) \\ &= -\int_{\widetilde{a}}^{\widetilde{b}} \bigl\langle D_\gamma D_\gamma V + \mathrm{R}(V, \dot\gamma, \dot\gamma), W\bigr\rangle_g + \bigl\langle D_\gamma V(t), W(t)\bigr\rangle_g \Big|_{t=\widetilde{a}}^{t=\widetilde{b}}.\end{aligned}$$

Summing up the terms over all such subintervals, gives the claimed description of $\mathrm{I}_\gamma(V, W)$ (check!). $\square$

3.5.2 Jacobi fields

The vector field of a variation of a geodesic through geodesics satisfies a corresponding differential equation, the Jacobi equation. Conversely, vector fields along curves that satisfy this equation are called Jacobi fields.

Here, we develop basic properties of Jacobi fields.

Proposition 3.5.7 (Jacobi equation). *Let (M, g) be a Riemannian manifold, let $\gamma\colon [a, b] \longrightarrow M$ be a goedesic, and let V be the variation field of a variation of γ through geodesics. Then,*

$$D_\gamma D_\gamma V + \mathrm{R}(V, \dot\gamma, \dot\gamma) = 0.$$

Proof. Let $t \in [a, b]$. Moreover, let $G\colon (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$ be a variation of γ through geodesics with variation field V. We write $S := \partial_{①} G$ and $T := \partial_{②} G$. For all $s \in (-\varepsilon, \varepsilon)$, we have $(D_{G(s,\,\cdot\,)} T)(s, t) = 0$, because $G(s,\,\cdot\,)$ is a geodesic and $T = \partial_{②} G$. Hence, $(D_{G(\,\cdot\,,t)} D_{G(0,\,\cdot\,)} T)(0, t) = 0$. Therefore, we obtain from the symmetry lemmas (Proposition 3.2.4, Proposition 3.5.3) that

$$\begin{aligned} D_\gamma D_\gamma V(t) + \mathrm{R}\bigl(V(t), \dot\gamma(t), \dot\gamma(t)\bigr) &= (D_{G(0,\,\cdot\,)} D_{G(0,\,\cdot\,)} S)(0, t) + \mathrm{R}\bigl(V(t), \dot\gamma(t), \dot\gamma(t)\bigr) \\ &= (D_{G(0,\,\cdot\,)} D_{G(\,\cdot\,,t)} T)(0, t) + \mathrm{R}\bigl(V(t), \dot\gamma(t), \dot\gamma(t)\bigr) && \text{(Proposition 3.2.4)} \\ &= (D_{G(\,\cdot\,,t)} D_{G(0,\,\cdot\,)} T)(0, t) && \text{(Proposition 3.5.3)} \\ &= 0, && \text{(see above)}\end{aligned}$$

as desired. $\square$

Study note. Before continuing to read on, you should compare the Jacobi equation with the explicit description of the index form (Proposition 3.5.6).

Definition 3.5.8 (Jacobi field)**.** Let (M, g) be a Riemannian manifold and let γ be a geodesic on M. A vector field V along γ satisfying the Jacobi equation

$$D_\gamma D_\gamma V + \mathrm{R}(V, \dot\gamma, \dot\gamma) = 0$$

is a *Jacobi field along* γ.

Proposition 3.5.9 (existence and uniqueness of Jacobi fields)**.** *Let* (M, g) *be a Riemannian manifold, let* $\gamma\colon [a,b] \longrightarrow M$ *be a geodesic. Let* $t \in [a,b]$ *and let* $v, w \in \mathrm{T}_{\gamma(t)} M$*. Then, there exists a unique Jacobi field* V *along* γ *with*

$$V(t) = v \quad \textit{and} \quad D_\gamma V(t) = w.$$

In particular: The set of Jacobi fields along γ *is an* $\mathbb{R}$*-vector subspace of* $\Gamma(\mathrm{T}\,M|\gamma)$ *of dimension* $2 \cdot \dim M$*.*

Proof. The second claim follows directly from the first claim (check!). For the first claim, it suffices to realise that Jacobi fields are exactly the solutions of a second order *linear* ordinary differential equation and that the conditions on $V(t)$ and $D_\gamma V(t)$ are two initial values of order 0 and 1, respectively.

More concretely, the Jacobi equation can be written as (check!)

$$\forall_{i \in \{1,\dots,n\}} \quad \ddot V^i + \sum_{j,k,\ell \in \{1,\dots,n\}} \mathrm{R}_{jk\ell}{}^i \circ \gamma \cdot V^j \cdot \dot\gamma^k \cdot \dot\gamma^\ell = 0$$

where $n := \dim M$, and $(V^1, \dots, V^n)$ are the coefficients of the desired vector field V with respect to an orthonormal parallel orthonormal frame along γ (which is also used to define the coefficients $\mathrm{R}_{jk\ell}{}^i$). $\square$

Example 3.5.10 (Jacobi fields)**.** Variation fields of variations through geodesics along geodesics are Jacobi fields (Proposition 3.5.7).

These are the only examples. The argument works as in the proof of Proposition 3.2.7: If $\gamma\colon [a,b] \longrightarrow M$ is a geodesic (with $0 \in [a,b]$) in a Riemannian manifold (M, g) and if V is a Jacobi field along γ, then we consider

$$G\colon (s,t) \longmapsto \exp_{\eta(s)}\bigl(t \cdot W(s)\bigr)$$

on a sufficiently thin rectangle; here, we choose a smooth curve η with $\dot\eta(0) = V(0)$ and a vector field W along η with $W(0) = \dot\gamma(0)$ and $D_\eta W(0) = D_\gamma V(0)$. Then, G has the following properties (check!):

- G defines a variation of γ;
- for each s, the curve $G(s, \,\cdot\,)$ is a geodesic;
- $\partial_{①} G(0,0) = V(0)$;

- $D_\gamma \partial_{①} G(0,0) = D_\gamma V(0)$.

Therefore, the uniqueness of Jacobi fields (Proposition 3.5.9) shows that V is the variation field of this variation.

Proposition 3.5.11 (normal Jacobi fields). *Let (M,g) be a Riemannian manifold, let $\gamma\colon I \longrightarrow M$ be a geodesic on M on an interval I with non-empty interior, and let V be a Jacobi field along γ. Then, the following are equivalent:*

1. *The Jacobi field V is normal along γ.*
2. *There exist two points in I at which V is normal to $\dot\gamma$.*
3. *There exists a $t \in I$ with $V(t) \perp \dot\gamma(t)$ and $D_\gamma V(t) \perp \dot\gamma(t)$.*

Proof. We consider the function $f := \langle V, \dot\gamma\rangle_g \colon I \longrightarrow \mathbb{R}$, measuring the part of V tangential to $\dot\gamma$. By elementary analysis, it suffices to prove the following properties on f (check!):

① The function f is affine linear.

② For all $t \in I$, we have $f'(t) = \big\langle D_\gamma V(t), \dot\gamma(t)\big\rangle_g$.

Ad ②. We have

$$\begin{aligned} f' &= \langle V, \dot\gamma\rangle_g' \\ &= \langle D_\gamma V, \dot\gamma\rangle_g + 0. && \text{(compatibility; } \gamma \text{ is a geodesic)} \end{aligned}$$

Ad ①. It suffices to show that $f'' = 0$ (check!). We have

$$\begin{aligned} f'' &= \langle V, \dot\gamma\rangle_g'' \\ &= \langle D_\gamma D_\gamma V, \dot\gamma\rangle_g + 0 && \text{(②; compatibility; } \gamma \text{ is a geodesic)} \\ &= -\big\langle \mathrm{R}(V,\dot\gamma,\dot\gamma), \dot\gamma\big\rangle_g && \text{(Jacobi equation)} \\ &= -\operatorname{Rm}(V,\dot\gamma,\dot\gamma,\dot\gamma) \\ &= 0. && \text{(Proposition 2.4.9)} \end{aligned}$$

Thus, f is affine linear. □

3.5.3 Conjugate points

Geometrically, as can be seen on round spheres, we expect that non-minimising geodesics are related to non-injectivity of the exponential map. The actual relation between non-minimising behaviour and non-injectivity of the exponential map is a bit less direct and involves a detour through conjugate points – a notion built on Jacobi fields.

Definition 3.5.12 (conjugate points)**.** Let (M, g) be a Riemannian manifold, let $x, y \in M$ with $x \neq y$, and let γ be a geodesic on M from x to y. Then y is *conjugate to x along γ* if there exists a non-zero Jacobi field V along γ that is 0 at x and y.

Quick check 3.5.13. In the situation of Definition 3.5.12, is the Jacobi field V normal to γ ?

Yes No

Proposition 3.5.14 (conjugate points and the exponential map)**.** *Let (M, g) be a Riemannian manifold, let $x \in M$, let $v \in \operatorname{Exp}_x$, and let $y := \exp_x v$. Then the following are equivalent:*

1. *The map $\exp_x$ is a local diffeomorphism on a neighbourhood of v.*
2. *The point y is* not *conjugate to x along* $\operatorname{geod}_{x,v}$.

Proof. By the inverse function theorem (and linear algebra on finite-dimensional vector spaces), the first condition is equivalent to:

1'. The linear map $\mathrm{d}_v \exp_x$ is injective.

The idea is to realise $\mathrm{d}_v \exp_x$ via values of Jacobi fields along $\operatorname{geod}_{x,v}$ and thus to connect condition 1' to the conjugacy property.

In the rest of this proof, we use the canonical identification $\mathrm{T}_v(\mathrm{T}_x M) \cong_{\mathbb{R}} \mathrm{T}_x M$. For $w \in \mathrm{T}_x M$ and a sufficiently small $\varepsilon > 0$ (check!), we consider the map

$$\begin{aligned} G \colon (-\varepsilon, \varepsilon) \times [0,1] &\longrightarrow M \\ (s,t) &\longmapsto \exp_x t \cdot (v + s \cdot w). \end{aligned}$$

By construction, G is a variation of $\gamma := \operatorname{geod}_{x,v}|_{[0,1]}$ through geodesics. Therefore, $V \colon t \mapsto \partial_{①} G(0,t)$ is a Jacobi field along γ (Example 3.5.10). Furthermore, we have

$$\begin{aligned} V(1) &= \partial_{①} G(0,1) = \frac{d}{ds}\Big|_{s=0} \exp_x(v + s \cdot w) = \mathrm{d}_v \exp_x(w) \\ V(0) &= \partial_{①} G(0,0) = 0 \\ D_\gamma V(0) &= w. \end{aligned}$$

Ad 2. $\Longrightarrow$ 1$'$. Let $\mathrm{d}_v \exp_x$ be *not* injective; i.e., there exists a $w \in \mathrm{T}_x M \setminus \{0\}$ with $\mathrm{d}_v \exp_x(w) = 0$. Then the Jacobi field V constructed above is non-zero (because) and witnesses that y is conjugate to x along γ.

Ad 1$'$. $\Longrightarrow$ 2. Let $\mathrm{d}_v \exp_x$ be injective. The uniqueness of Jacobi fields (Propsoition 3.5.9) shows that the above construction (with varying w) exhausts all Jacobi fields along γ that vanish at x. Thus, if $\mathrm{d}_v \exp_x$ is injective, then there is *no* non-zero Jacobi field along γ that vanishes at x and y. Thus, y is *not* conjugate to x along γ. □

Theorem 3.5.15 (geodesics are not minimising past conjugate points). *Let (M, g) be a Riemannian manifold, let $x, y \in M$, and let $\gamma \colon [a, b] \longrightarrow M$ be a geodesic from x to y. Moreover, let $t \in (a, b)$ have the property that $\gamma(t)$ is conjugate to x along γ.*

1. *Then, there exists a proper normal piecewise smooth vector field V along γ with $\mathrm{I}_\gamma(V, V) < 0$.*

2. *In particular: The curve γ is* not *minimising.*

Proof. The second part follows from the first part and the index criterion (Corollary 3.5.5).

For the first part, we argue as follows: Without loss of generality, we may assume that $a = 0$ and that γ has unit speed (check!). We write $z := \gamma(t)$.

As z is conjugate to x along γ, there exists a non-trivial Jacobi field V along $\gamma|_{[0,t]}$ with

$$V(0) = 0 = V(t).$$

Moreover, V is normal (Quick check 3.5.13). We now extend V to a piecewise smooth normal vector field along γ:

$$\begin{aligned} \widetilde{V} \colon [0, b] &\longrightarrow \mathrm{T}\, M \\ s &\longmapsto \begin{cases} V(s) & \text{if } s \in [0, t] \\ 0 & \text{if } s \in [t, b]. \end{cases} \end{aligned}$$

In order to find a proper normal vector field along γ with negative index, we deform $\widetilde{V}$ suitably: Using bump functions, we can construct a smooth proper normal vector field W along γ with

$$W(t) = \Delta_t D_\gamma \widetilde{V} = -D_\gamma V(t)$$

(check!). Moreover, by the uniqueness of Jacobi fields (Proposition 3.5.9) and non-triviality of V, we know that $D_\gamma V(t) \neq 0$; hence, $W(t) \neq 0$.

For $\varepsilon \in \mathbb{R}_{>0}$, we consider now the deformed vector field

$$\widetilde{V}_\varepsilon := \widetilde{V} + \varepsilon \cdot W.$$

By construction, $\widetilde{V}_\varepsilon$ is a proper normal piecewise smooth vector field along γ. It remains to compute the index: Because the index form is bilinear and symmetric (check!), we have

$$\mathrm{I}_\gamma(\widetilde{V}_\varepsilon, \widetilde{V}_\varepsilon) = \mathrm{I}_\gamma(\widetilde{V}, \widetilde{V}) + 2 \cdot \varepsilon \cdot \mathrm{I}_\gamma(\widetilde{V}, W) + \varepsilon^2 \cdot \mathrm{I}_\gamma(W, W).$$

The first two terms can be simplified as follows: We have

$$
\begin{aligned}
\mathrm{I}_\gamma(\widetilde{V},\widetilde{V}) &= -\int_0^b \bigl\langle D_\gamma D_\gamma \widetilde{V} + \mathrm{R}(\widetilde{V},\dot\gamma,\dot\gamma),\widetilde{V}\bigr\rangle_g - \bigl\langle \Delta_t D_\gamma \widetilde{V},\widetilde{V}(t)\bigr\rangle_g && \text{(Proposition 3.5.6)}\\
&= 0+0 && \text{(Jacobi equation on } [0,t] \text{ and } [t,b];\ \widetilde{V}(0)=0)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{I}_\gamma(\widetilde{V},W) &= -\int_0^b \bigl\langle D_\gamma D_\gamma \widetilde{V} + \mathrm{R}(\widetilde{V},\dot\gamma,\dot\gamma),W\bigr\rangle_g - \bigl\langle \Delta_t D_\gamma \widetilde{V},W(t)\bigr\rangle_g \\
&= 0 - \bigl\langle -D_\gamma V(t), -D_\gamma V(t)\bigr\rangle_g && \text{(see above; by construction)}\\
&= -\bigl\|D_\gamma V(t)\bigr\|_g^2.
\end{aligned}
$$

Therefore, we obtain

$$
\mathrm{I}_\gamma(\widetilde{V}_\varepsilon,\widetilde{V}_\varepsilon) = -2\cdot\varepsilon\cdot\bigl\|D_\gamma V(t)\bigr\|_g^2 + \varepsilon^2\cdot \mathrm{I}_\gamma(W,W).
$$

Because $D_\gamma V(t) \neq 0$, there exists a (small) $\varepsilon \in \mathbb{R}_{>0}$ with $\mathrm{I}_\gamma(\widetilde{V}_\varepsilon,\widetilde{V}_\varepsilon) < 0$. □

Caveat 3.5.16. There exist geodesics without conjucate points that are *not* globally minimising. For example, this happens on the round circle and on the flat cylinder $\mathbb{R}\times\mathbb{S}^1$ (check!).

Study note (summary)**.** Write a summary of Chapter 3, keeping the following questions in mind:

- How are (locally) minimising curves and geodesics related?
- How are the exponential map, radial geodesics, and normal coordinates used in this context?
- Which role is played by the first and second variation formula?
- What are Jacobi fields and what are they good for?
- What are key identities that are used over and over again in computations?

4

Curvature: Local vs. global

Finally, we collected enough tools to prove meaningful results, linking local properties of Riemannian manifolds (i.e., curvature conditions) to their global shape (i.e., topological invariants such as the fundamental group).

More specifically, we will show the following classical results for complete manifolds:

- Manifolds of positive Ricci curvature satisfy a uniform diameter bound (and thus have finite fundamental group).
- Manifolds of non-positive sectional curvature are aspherical: their universal covering is contractible.
- Manifolds of constant sectional curvature are quotients of the model spaces.

We will prove these results via Jacobi fields and comparison estimates.

Furthermore, we will take another step in the study of the interaction between geometry and fundamental groups: We will quickly discuss the Švarc-Milnor lemma and its applications.

Overview of this chapter.

Running example. model spaces and their quotients

4.1 Local-global results

Local-global results in differential geometry concern the global consequences of local curvature constraints. The prototypical example is the Gauß-Bonnet theorem:

Theorem 4.1.1 (Gauß-Bonnet [18, Theorem 9.7]). *Let M be an oriented closed connected surface. Then, for every Riemannian metric g on M, we have*

$$\int_M K \, d\mu_M = 2 \cdot \pi \cdot \chi(M).$$

Here, $K \colon M \longrightarrow \mathbb{R}$ denotes the Gaußian curvature of M, i.e., for $x \in M$, the value $K(x)$ is the sectional curvature of the tangent plane $\mathrm{T}_x M$; moreover, $\chi(M)$ denotes the Euler characteristic of M, i.e., the alternating sum of the number of vertices, edges, and triangles in a triangulation of M.

Why is this a local-global result? The local curvature information is assembled into a global, topological, invariant (the Euler characteristic). For example, as the Euler characteristic of the 2-sphere is positive, the Euler-characteristic of the torus is zero, and the Euler characteristic of higher genus surfaces is negative, one can conclude from the Gauß-Bonnet theorem that

- the 2-sphere does not admit a Riemannian metric of (everywhere) non-positive sectional curvature, that
- the torus neither admits a Riemannian metric of (everywhere) positive sectional curvature nor one of (everywhere) negative sectional curvature, and that
- the surfaces of higher genus do not admit Riemannian metrics of (everywhere) non-negative sectional curvature.

In this course, we will *not* prove the Gauß-Bonnet theorem. Instead, we will focus on statements that work in general dimensions and have similar consequences (even though they will be far less explicit than the Gauß-Bonnet theorem). Most of these results will be based on *comparison* results: Curvature bounds will be turned into bounds on Jacobi fields, which in turn will give bounds on other geometric quantities.

4.2 Constant sectional curvature, locally

Before delving into comparison results, we need to understand the base case, namely the local uniqueness of Riemannian manifolds with constant sectional curvature: If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds that have the same *constant* sectional curvature, then (M_1, g_1) and (M_2, g_2) are locally isometric. The proof will also give explicit local descriptions.

The idea is to use Jacobi fields: Jacobi fields describe how variations of geodesics evolve. In the case of constant sectional curvature, Jacobi fields can be described explicitly and in combination with the Gauß lemma, we obtain a full, local, description of the underlying Riemannian metric.

Proposition 4.2.1 (Jacobi fields in constant sectional curvature). *Let (M, g) be a Riemannian manifold of constant sectional curvature $c \in \mathbb{R}$ and let $\gamma\colon I \longrightarrow M$ be a unit speed geodesic on M with $0 \in I$. Then, the normal Jacobi fields along γ with $V(0) = 0$ are exactly the vector fields V along γ for which there exists a parallel normal vector field E along γ with*

$$V = u \cdot E,$$

where u is the generalised sine function *associated with c (and $R := 1/\sqrt{|c|}$):*

$$\begin{aligned} u\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} t & \text{if } c = 0 \\ R \cdot \sin \frac{t}{R} & \text{if } c > 0 \\ R \cdot \sinh \frac{t}{R} & \text{if } c < 0. \end{cases} \end{aligned}$$

Remark 4.2.2 (hyperbolic sine function). The hyperbolic (co)sine function plays the same role for the standard hyperbola as the usual (co)sine functions for the unit circle. As we will be mainly interested in analytic properties of the hyperbolic sine function, we introduce it as follows: The *hyperbolic sine* is defined as:

$$\begin{aligned} \sinh\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{e^x - e^{-x}}{2} \end{aligned}$$

The hyperbolic sine function satisfies the same differential equation as the sine function – except for an inverted sign:

$$\sinh'' - \sinh = 0.$$

Proof of Proposition 4.2.1. From the existence and uniqueness of Jacobi fields (Proposition 3.5.9), we know that the set of Jacobi fields along γ that are normal and vanish at 0 is an $\mathbb{R}$-vector space of dimension $n - 1$ (where n is the dimension of M).

As parallel vector fields are determined by their initial vector, the normality condition shows that the space of parallel normal vector fields along γ has also dimension $n - 1$.

Therefore, it suffices to show that the vector fields of the claimed form are Jacobi fields.

Let E be a parallel normal vector field along γ, let u be the generalised sine function associated with c, and let $V := u \cdot E$. Then

$$u'' + c \cdot u = 0$$

(check!). In combination with the fact that $\sec = c$, we obtain

$$\begin{aligned}
D_\gamma D_\gamma V + \mathrm{R}(V, \dot\gamma, ;\gamma) &= D_\gamma D_\gamma V + c \cdot \bigl(\langle \dot\gamma, \dot\gamma \rangle_g \cdot V - \langle V, \dot\gamma \rangle_g \cdot \dot\gamma\bigr) && (\text{because } \sec = c;\ \text{Exercise}) \\
&= D_\gamma D_\gamma V + c \cdot (1 \cdot V - 0) && (\gamma \text{ has unit speed; } V \text{ is normal to } \gamma) \\
&= u'' \cdot E + c \cdot u \cdot E && (\text{because}\qquad) \\
&= 0 \cdot E && (u \text{ is the generalised sine function for } c) \\
&= 0.
\end{aligned}$$

In other words, $V = u \cdot E$ satisfies the Jacobi equation along γ. Moreover, $V(0) = 0$ and V is normal. ☐

Study note. The above proof of Proposition 4.2.1 is a typical example of a hindsight proof. In discovery mode, one would proceed as follows: Starting from the Jacobi equation, one might make the ansatz $u \cdot E$, where E is a parallel normal vector field along γ and where u is a smooth function. Then the Jacobi equation for such a vector field translates into the second order differential equation

$$u'' + c \cdot u = 0,$$

which is solved by the generalised sine functions. A dimension count reveals that indeed every solution has to have the form of the ansatz. However, such proofs are harder to write up with a high degree of rigor and precision. Therefore, we preferred the above hindsight version of the proof.

Proposition 4.2.3 (Jacobi fields along radial geodesics)**.** *Let (M, g) be a Riemannian manifold of dimension n, let $x \in M$, let $U \subset M$ be a normal neighbourhood of x, and let $(E_j)_{j \in \{1,\dots,n\}}$ be the coordinate frame associated with some normal coordinates on U, centred at x. Let $\gamma \colon [0, b] \longrightarrow U$ be a radial geodesic starting at x and let $v = \sum_{j=1}^n v^j \cdot E_j(x) \in \mathrm{T}_x M$. Then the Jacobi field V along γ with*

$$V(0) = 0 \quad \textit{and} \quad D_\gamma V(0) = v$$

is given by

$$\begin{aligned}
V \colon [0, b] &\longrightarrow \mathrm{T}\, M \\
t &\longmapsto t \cdot \sum_{j=1}^n v^j \cdot E_j\bigl(\gamma(t)\bigr).
\end{aligned}$$

Proof. A straightforward computation via the expression of radial geodesics in normal coordinates shows that V satisfies the initial conditions (Exercise).

Moreover, V is a Jacobi field along γ: Indeed, V is the variation field of the variation of γ through geodesics (for a suitable choice of $(w^1, \dots, w^n)$), given in local coordinates by

$$(s,t) \longmapsto t \cdot (v^1 + s \cdot w^1, \dots, v^n + s \cdot w^n)$$

(Exercise). Uniqueness of V holds by the general uniqueness of Jacobi fields with given initial values (Proposition 3.5.9). $\square$

Theorem 4.2.4 (constant sectional curvature, locally). *Let (M,g) be a Riemannian manifold of constant sectional curvature $c \in \mathbb{R}$. Let $x \in M$, let $U \subset M$ be a normal neighbourhood centred at x, and let $\varphi\colon U \longrightarrow U'$ be a normal coordinate chart centred at x. Moreover, let $\overline{g}$ denote the φ-pullback of the Euclidean Riemannian metric on U' to U. For all $y \in U \setminus \{x\}$ and all $v \in \mathrm{T}_y M$, we have*

$$\|v\|_g^2 = \|v^\perp\|_{\overline{g}}^2 + \frac{1}{\varrho^2(y)} \cdot u^2\bigl(\varrho(y)\bigr) \cdot \|\overline{v}\|_{\overline{g}}^2,$$

where $v = \overline{v} + v^\perp$ is the orthogonal decomposition of v into the part $\overline{v}$ tangential to the geodesic sphere through y and the part $v^\perp$ in the direction of $\partial/\partial\varrho$, and where u is the generalised sine function associated with c.

Proof. Let $v \in \mathrm{T}_y M$. Then $v = \overline{v} + v^\perp$ indeed is an orthogonal descomposition by . Therefore, we have

$$\|v\|_g^2 = \|\overline{v}\|_g^2 + \|v^\perp\|_g^2.$$

We now treat both terms separately:

- We have $\|v^\perp\|_g^2 = \|v^\perp\|_{\overline{g}}^2$ because $\partial/\partial\varrho$ has unit length with respect to g (Remark 3.2.18) and $\overline{g}$ (check!).
- Thus, it remains to investigate $\|\overline{v}\|_g^2$: We use Jacobi fields; Jacobi fields describe the spread of geodesics, whence a suitable Jacobi field should compute this contribution. Let $(E_1, \dots, E_n)$ be the coordinate frame associated with φ, let $(v^1, \dots, v^n)$ be the corresponding coordinates of $\overline{v}$, let $r := \varrho(y)$, and let $\gamma\colon [0,r] \longrightarrow M$ be the unit speed radial geodesic from x to y. Then

$$\begin{aligned} V\colon [0,r] &\longrightarrow \mathrm{T} M \\ t &\longmapsto \frac{t}{r} \cdot \sum_{j=1}^n v^j \cdot E_j \end{aligned}$$

  is a Jacobi field along γ with

$$V(0) = 0 \quad \text{and} \quad V(r) = \frac{1}{r} \cdot \overline{v}$$

(Proposition 4.2.3). As $\overline{v}$ is normal to γ at r, the Jacobi field V is normal to γ (Proposition 3.5.11). By the explicit description of normal Jacobi fields in constant sectional curvature (Proposition 4.2.1), there exists a parallel normal vector field E along γ such that

$$V = u \cdot E.$$

Therefore, we obtain

$$\begin{aligned}
\|\overline{v}\|_g^2 &= \big\|V(r)\big\|_g^2 = \big|u(r)\big|^2 \cdot \big\|E(r)\big\|_g^2 \\
&= \big|u(r)\big|^2 \cdot \big\|E(0)\big\|_g^2 && \text{(parallel vector fields have constant length)} \\
&= \big|u(r)\big|^2 \cdot \big\|u'(0) \cdot E(0)\big\|_g^2 && \text{(because } u \text{ is a generalised sine function)} \\
&= \big|u(r)\big|^2 \cdot \big\|D_\gamma V(0)\big\|_g^2 && \text{(construction of } V \text{ and } E \text{ is parallel)} \\
&= \big|u(r)\big|^2 \cdot \big\|D_\gamma V(0)\big\|_{\overline{g}}^2 && \text{(at } x \text{ the inner products } g \text{ and } \overline{g} \text{ agree)} \\
&= \big|u(r)\big|^2 \cdot \Big\|\frac{1}{r} \cdot \overline{v}\Big\|_{\overline{g}}^2. && \text{(Proposition 4.2.3)}
\end{aligned}$$

Summing up both terms proves the theorem. □

Study note. Can you express the local description of Riemannian metrics of constant sectional curvature in Theorem 4.2.4 as a warped product?

Corollary 4.2.5 (local uniqueness of constant sectional curvature metrics)**.** *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds that have the same constant sectional curvature. Then (M_1, g_1) and (M_2, g_2) are locally isometric.*

In particular, they are both locally isometric to the model space with the same constant sectional curvature.

Proof. This is a direct consequence of Theorem 4.2.4 and polarisation (check!). The second claim uses that the model spaces have constant sectional curvature and that each real number can be realised as the constant sectional curvature of a model space (Theorem 2.5.11). □

4.3 Analytic and geometric comparison theorems

Analytically, the key observation is that upper bounds on sectional curvature lead to lower bounds on Jacobi fields. Geometrically, these lower bounds on Jacobi fields translate into estimates for conjugate points and into comparison results for Riemannian metrics.

The underlying analytic tool is the Sturm comparison theorem, which pulls a pointwise estimate out of the differential (in)equality for the generalised sine functions:

Theorem 4.3.1 (Sturm comparison theorem)**.** *Let $T \in \mathbb{R}_{>0}$ and let $a\colon [0,T) \longrightarrow \mathbb{R}$ be a function. Moreover, let $u, v\colon [0,T) \longrightarrow \mathbb{R}$ be differentiable functions with $u|_{(0,T)} > 0$ that are twice differentiable on $(0,T)$ and satisfy*

$$\begin{aligned} &u'' + a \cdot u = 0 \\ &v'' + a \cdot v \geq 0 \\ &u(0) = v(0) = 0 \\ &u'(0) = v'(0) > 0. \end{aligned}$$

Then:

$$\forall_{t \in [0,T)} \quad v(t) \geq u(t).$$

Proof. We consider the quotient function

$$\begin{aligned} f\colon [0,T) &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} \frac{v(t)}{u(t)} & \text{if } t > 0 \\ 1 & \text{if } t = 0. \end{cases} \end{aligned}$$

The function f is continuous in view of l'Hospital's rule and the initial values of u and v (check!); moreover, f is differentiable on $(0,T)$. Because $f(0) = 1$, it suffices to show that $f'(t) \geq 0$ for all $t \in (0,T)$.

On $(0,T)$, we have

$$f' = \frac{v' \cdot u - v \cdot u'}{u^2}.$$

As the denominator is non-negative, we only need to show that $g := v' \cdot u - v \cdot u'$ is non-negative on $(0,T)$. Because of $g(0) = 0 - 0 = 0$, we can once again use the derivative: It suffices to show that $g' \geq 0$ on $(0,T)$. The differential (in)equalities for u and v show that

$$\begin{aligned} g' &= (v' \cdot u - v \cdot u')' \\ &= v'' \cdot u + v' \cdot u' - v' \cdot u' - v \cdot u'' \\ &= v'' \cdot u - v \cdot u'' \\ &\geq -a \cdot v \cdot u - v \cdot a \cdot u && \text{(differential (in)equality)} \\ &= 0. \end{aligned}$$

This completes the proof. □

Theorem 4.3.2 (Jacobi field comparison theorem)**.** *Let (M,g) be a Riemannian manifold, let $c \in \mathbb{R}$, and let $\sec \leq c$ on M. Let $\gamma\colon I \longrightarrow M$ be a unit speed geodesic on M with $0 \in I$ and let V be a normal Jacobi field along γ with $V(0) = 0$. Then, for all $t \in I$, we have (with $R := 1/\sqrt{|c|}$):*

$$\|V(t)\|_g \geq \begin{cases} \|D_\gamma V(0)\|_g \cdot t & \textit{if } c = 0 \textit{ and } t \geq 0 \\ \|D_\gamma V(0)\|_g \cdot R \cdot \sin \frac{t}{R} & \textit{if } c > 0 \textit{ and } t \in [0, \pi \cdot R] \\ \|D_\gamma V(0)\|_g \cdot R \cdot \sinh \frac{t}{R} & \textit{if } c < 0 \textit{ and } t \geq 0. \end{cases}$$

Proof. The idea is to apply the Sturm comparison theorem (Theorem 4.3.1) on $[0, T_c)$ to $v := \|V\|_g$ and the generalised sine function u for c; here, $T_c := \infty$ if $c \leq 0$; and $T_c := \pi \cdot R$ for $c > 0$. We thus establish that the Sturm comparison theorem is applicable in this situation:

- The function u is twice differentiable on $\mathbb{R}$, satisfies the differential equation $u'' + c \cdot u = 0$ and $u(0) = 0$ as well as $u'(0) = 1 > 0$: These are standard properties of the generalised sine functions.
- Without loss of generality, we may assume that $D_\gamma V(0) = 1$ because .
- The function v is smooth at all points $t \in I$ with $V(t) \neq 0$ (check!).
- Moreover, v is differentiable at 0 with $v'(0) = 1 = u'(0)$: Indeed, we have

$$\begin{aligned} \lim_{t \to 0} \frac{v(t) - v(0)}{t} \lim_{t \to 0} \frac{\|V(t)\|_g - \|V(0)\|_2}{t} & \\ = \lim_{t \to 0} \frac{t \cdot \|W(t)\|_g}{t} & \quad \text{(for a suitable smooth vector field } W \text{ (Proposition 4.2.3))} \\ = \lim_{t \to \infty} \|W(t)\|_t = W(0) & \\ = D_\gamma V(0) & \quad \text{(Proposition 4.2.3)} \\ = 1. & \end{aligned}$$

- If $t \in I$ with $V(t) \neq 0$, then $v''(t) + c \cdot v(t) \geq 0$: Using compatibility of the Levi-Civita connection (Proposition 2.3.7) and the Jacobi equation, we obtain on $J := \{t \in I \mid V(t) \neq 0\}$ that

$$v' = \|V\|_g' = \bigl((\langle V, V\rangle_g)^{1/2}\bigr)' = \frac{1}{2} \cdot 2 \cdot \frac{\langle D_\gamma V, V\rangle_g}{\langle V, V\rangle_g^{1/2}} = \frac{\langle D_\gamma V, V\rangle_g}{\langle V, V\rangle_g^{1/2}}$$

and thus

$$\begin{aligned} v'' &= \frac{\langle D_\gamma D_\gamma V, V\rangle_g \cdot \langle V, V\rangle_g^{1/2} + \langle D_\gamma V, D_\gamma V\rangle_g \cdot \langle V, V\rangle_g^{1/2} - \langle D_\gamma V, V\rangle_g \cdot \langle V, V\rangle_g^{-1/2} \cdot \langle D_\gamma V, V\rangle_g}{\langle V, V\rangle_g} \\ &= \frac{\langle -\mathrm{R}(V, \dot\gamma, \dot\gamma), V\rangle_g}{\|V\|_g} + \frac{\|D_\gamma V\|_g^2 \cdot \|V\|_g^2 - \langle D_\gamma V, V\rangle_g^2}{\|V\|_g^3} && \text{(Jacobi equation)} \\ &\geq \frac{-\mathrm{Rm}(V, \dot\gamma, \dot\gamma, V)}{\|V\|_g}. && \text{(Cauchy-Schwarz inequality)} \end{aligned}$$

We now rewrite this term as sectional curvature: If $t \in J$, then $V(t) \perp \dot\gamma(t)$ and $V(t) \neq 0$ and $\|\dot\gamma(t)\|_g = 1$. Therefore, $\sec_{\gamma(t)}(V(t), \dot\gamma(t))$ is defined and

$$\begin{aligned} v''(t) &\geq -\sec_{\gamma(t)}\bigl(V(t), \dot\gamma(t)\bigr) \cdot \bigl\|V(t)\bigr\|_g && \text{(by definition of sec)} \\ &\geq -c \cdot \bigl\|V(t)\bigr\|_g && \text{(by the hypothesis } \sec \leq c\text{)} \\ &= -c \cdot v(t). \end{aligned}$$

It remains to argue why we can work on the whole interval. As $v'(0) = 1$, there exists an $\varepsilon \in (0, T_c)$ with $v|_{(0,\varepsilon)} > 0$. Thus, we may apply the Sturm comparison theorem (Theorem 4.3.1) on $[0, \varepsilon)$ to v and u. Therefore,

$$v|_{(0,\varepsilon)} \geq u_{(0,\varepsilon)} > 0.$$

By continuity, $v(\varepsilon) \geq u(\varepsilon) > 0$. Arguing in the same way, we can extend the interval $(0, \varepsilon)$ and we see that we have $v|_{(0,T_c)} \geq u|_{(0,T_c)} > 0$ (check!). Unfolding the definitions of v and u, this completes the proof. □

Corollary 4.3.3 (conjugate point comparison theorem). *Let (M, g) be a Riemannian manifold, let $c \in \mathbb{R}$, and let* $\sec \leq c$ *on M.*

1. *If $c \leq 0$, then M has* no *conjugate points along any geodesic.*
2. *If $c > 0$ and $R := 1/\sqrt{c}$, then the first conjugate point along any geodesic on M cannot occur before a distance of $\pi \cdot R$.*

Proof. This follows directly from the Jacobi field comparison theorem (Theorem 4.3.2). □

Corollary 4.3.4 (Riemannian metric comparison theorem). *Let (M, g) be a Riemannian manifold, let $c \in \mathbb{R}$, and let* $\sec \leq c$ *on M. Let U be a normal coordinate chart centred at x and let $y \in U$; if $c > 0$, then we additionally assume that $d_g(x, y) \leq \pi \cdot 1/\sqrt{c}$. Then, for all $v \in \mathrm{T}_y U$, we have*

$$\|v\|_g \geq \|v\|_{g_c},$$

where g_c is the Riemannian metric on U with constant sectional curvature c specified in Theorem 4.2.4.

Proof. We decompose $v = \overline{v} + v^\perp$ orthogonally into the part tangential to the geodesic sphere through y around the centre x of U and the radial part $v^\perp$, using the Gauß lemma (Proposition 3.2.19). Then

$$\|v\|_g^2 = \|\overline{v}\|_g + \|v^\perp\|_g.$$

On the one hand, $\|v^\perp\|_g = \|v^\perp\|_{g_c}$ (literally the same argument as in the proof of Theorem 4.2.4 applies).

For the summand $\|\overline{v}\|_g$, we proceed as follows: Let $\gamma\colon [0,r] \longrightarrow M$ denote the unit speed radial geodesic from x to y. Then there exists a Jacobi field V along γ with $V(0) = 0$ and $V(r) = \overline{v}$ (this can be shown as in the proof of Proposition 3.5.14 because $\exp_x$ is a local diffeomorphism on all of U; check!); this Jacobi field is normal (Proposition 3.5.11). Therefore, the Jacobi field comparison theorem (Theorem 4.3.2) shows (where u is the generalised sine function for c)

$$\begin{aligned}
\|\overline{v}\|_g &= \big\|V(r)\big\|_g \\
&\geq u(r) \cdot \big\|D_\gamma V(0)\big\|_g && \text{(Jacobi field comparison theorem)} \\
&= u(r) \cdot \big\|D_\gamma V(0)\big\|_{g_c}. && \text{(because } \qquad\qquad\qquad \text{)}
\end{aligned}$$

The given normal coordinates for g are also normal coordinates for g_c (check!). Thus, γ is also a radial geodesic with respect to g_c (check!) and V is a Jacobi field along γ with respect to g_c (check!). Therefore, in combination with Proposition 4.2.1, we obtain

$$\begin{aligned}
\|\overline{v}\|_g &\geq u(r) \cdot \big\|D_\gamma V(0)\big\|_{g_c} \\
&= \big\|V(r)\big\|_{g_c} \\
&= \|\overline{v}\|_{g_c}.
\end{aligned}$$

As $\overline{v}$ and $v^\perp$ are orthogonal with respect to g_c (check!), these contributions sum up to

$$\|v\|_g^2 \geq \|\overline{v}\|_{g_c} + \|v^\perp\|_{g_c} = \|v\|_{g_c}. \qquad \square$$

Remark 4.3.5 (spreading geodesics)**.** The previous comparison results can together with the second variation formula (Theorem 3.5.2) roughly be summarised as follows (Figure 4.1):

- In flat manifolds, geodesics emanating from the same point spread as in the Euclidean plane.
- In manifolds of uniformly negative sectional curvature, geodesics emanating from the same point spread further apart than in hyperbolic spaces.
- In manifolds of uniformly positive sectional curvature, geodesics emanating from the same point come back together as on sphere.

Outlook 4.3.6 (comparison geometry)**.** The comparison theorems are the starting point of comparison geometry [6], which is particularly well adapted to non-positive curvature. More precisely, one realised that the purely metric behaviour of geodesic triangle (and their comparison with geodesic triangles in Euclidean and hyperbolic space) already can be used to deduce many standard results on non-positively curved manifolds – without using analytic

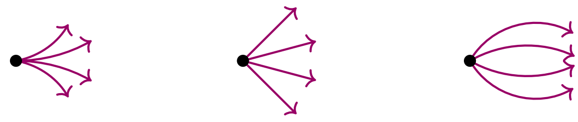


Figure 4.1.: Geodesic spread, schematically

tools such as curvature tensors! In this setting, one typically considers so-called CAT(0)-spaces (named after Cartan, Alexandrov, and Toponogov) or Gromov hyperbolic spaces [5].

Why is this metric approach useful? On the one hand, the metric approach clarifies which phenomena are of a metric nature and which require analytic input. On the other hand, the additional generality allows for many applications in other fields, e.g., in geometric group theory [8, 19].

4.4 Non-positive curvature

As a first application of the comparison results, we consider the case of non-positive curvature, i.e., the case of Riemannian manifolds whose sectional curvature is everywhere non-positive.

Caveat 4.4.1 ((non-)positive)**.** "Non-positive sectional curvature" is *not* the same as "not of positive sectional curvature". The latter only means that there exists a point with a tangent plane whose sectional curvature is not positive.

4.4.1 The Cartan-Hadamard theorem

A topological consequence of non-positive curvature is that all complete manifolds with non-positive sectional curvature are aspherical in a strong sense:

Theorem 4.4.2 (Cartan-Hadamard)**.** *Let (M, g) be a connected complete non-empty Riemannian manifold with* $\sec \leq 0$*.*

1. *If $x \in M$, then $\operatorname{Exp}_x = \operatorname{T}_x M$ and the exponential map $\exp_x \colon \operatorname{T}_x M \longrightarrow M$ is a covering map.*
2. *In particular: "The" universal covering space of M (with the induced smooth structure) is diffeomorphic to $\mathbb{R}^{\dim M}$. If M is simply connected, then M is diffeomorphic to $\mathbb{R}^{\dim M}$ via $\exp_x$.*

Proof. Let $n := \dim M$.

Ad 1. By completeness, $\mathrm{Exp}_x = \mathrm{T}_x M$ (Theorem 3.3.4). As $\sec \leq 0$, the conjugate point comparison theorem (Theorem 4.3.3) shows: The point x has *no* conjugate points along any geodesic. Therefore, $\exp_x$ is a local diffeomorphism on all of $\mathrm{T}_x M$ (Proposition 3.5.14).

Let $\widetilde{g} := \exp_x^* g$ be the induced Riemannian metric on $\mathrm{T}_x M$ (Proposition 1.3.4). In particular, $\exp_x \colon (\mathrm{T}_x M, \widetilde{g}) \longrightarrow (M, g)$ is a local isometry. Moreover, $(\mathrm{T}_x M, \widetilde{g})$ is complete (Exercise). Thus, $\exp_x \colon \mathrm{T}_x M \longrightarrow M$ is a covering map (Exercise).

Ad 2. By the first part, $\exp_x \colon \mathrm{T}_x M \longrightarrow M$ is a covering map. Moreover, $\mathrm{T}_x M$ is simply connected. Therefore, $\exp_x \colon \mathrm{T}_x M \longrightarrow M$ is "the" universal covering of M.

As seen in the proof of the first part, the exponential map $\exp_x \colon \mathrm{T}_x M \longrightarrow M$ is a local diffeomorphism. Thus, the standard smooth structure on $\mathrm{T}_x M$ coincides with the smooth structure on $\mathrm{T}_x M$ induced by the universal covering map $\exp_x$.

In particular, the universal covering space of M is diffeomorphic to $\mathbb{R}^n$. □

Outlook 4.4.3 (aspherical manifolds)**.** A connected manifold is *aspherical* if tis universal covering is contractible (equivalently, if the homotopy groups in degree ≥ 2 are all trivial).

The Cartan-Hadamard theorem (Theorem 4.4.2) shows: All complete connected non-empty Riemannian manifolds of non-positive sectional curvature are aspherical manifolds: Their universal covering is even diffeomorphic to a Euclidean space!

In particular, compact Riemannian manifolds with non-positive sectional curvature have infinite fundamental group [20, Example 4.2.9].

On the other hand, it is known that there exist *exotic* aspherical manifolds, i.e., closed aspherical manifolds whose universal covering is *not* homeomorphic to a Euclidean space [7].

Closed aspherical manifolds are objects of active research and play an important role in topological and geometric rigidity.

Example 4.4.4. Let $n \in \mathbb{N}_{\geq 2}$. Then $\mathbb{S}^n$ and $\mathbb{R}P^n$ do *not* admit Riemannian metrics of non-positive sectional curvature (because

).

Corollary 4.4.5 (volume growth in negative curvature)**.** *Let (M, g) be a simply connected complete Riemannian manifold with $\sec \leq c < 0$, where $c \in \mathbb{R}_{<0}$ and let $x \in M$. Then the volume growth function $\varrho_x^{M,g}$ at x grows at least exponentially.*

Proof. By rescaling, we may assume that $c = -1$ (check!). On the one hand, we already know that hyperbolic spaces of dimension $n := \dim M \geq 2$ have (at least) exponential growth (Example 1.5.22).

On the other hand, we can compare (M, g) with $\mathbb{H}^n$. By the Cartan-Hadamard theorem (Theorem 4.4.2), M is a normal neighbourhood of x. We can then use the Riemannian metric comparison theorem (Corollary 4.3.4) to show that $\varrho_x^{M,g}$ dominates the volume growth of the corresponding hyperbolic space (Exercise). This involves turning a norm estimate into a determinant estimate, which can be achieved through the principal axis theorem or the Courant minimax principle (Exercise). $\square$

4.4.2 Constant non-positive sectional curvature

Theorem 4.4.6 (manifolds of constant non-positive sectional curvature). *Let (M, g) be complete, simply connected, non-empty Riemannian manifold with constant sectional curvature c and let $n := \dim M$.*

1. *If $c = 0$, then (M, g) is isometric to $\mathbb{R}^n$ with the Euclidean Riemannian metric.*
2. *If $c < 0$, then (M, g) is isometric to $\mathbb{H}^n(R)$ with $R := 1/\sqrt{|c|}$.*

Proof. Let $x \in M$. By the Cartan-Hadamard theorem (Theorem 4.4.2), the exponential map $\exp_x \colon \mathrm{T}_x M \longrightarrow M$ is a diffeomorphism. Thus, M is a normal neighbourhood of x.

Also, $\mathbb{R}^n$ (with the Euclidean Riemannian metric) is a normal neighbourhood of each point in $\mathbb{R}^n$ and $\mathbb{H}^n(R)$ is a normal neighbourhood of each point in $\mathbb{R}^n$.

Now the local description of constant sectional curvature metrics (Theorem 4.2.4) and the curvature computation for the model spaces (Theorem 2.5.11) completes the proof. $\square$

Example 4.4.7 (surfaces of higher genus). Let $g \in \mathbb{N}_{\geq 2}$. Then the universal covering of the oriented closed connected surface of genus 2 is diffeomorphic to $\mathbb{R}^2$; there are many ways of seeing this, one of them is via Theorem 4.4.6 and elementary hyperbolic geometry:

It is possible to find a geodesic $4g$-gon D in $\mathbb{H}^2$, all of whose edge lengths are equal and whose hyperbolic angles at the vertices are all equal to $2\pi/4g$. Glueing the edges of D according to the $a_1 b_1 a_1^{-1} b_1 \dots a_g b_g a_g^{-1} b_g{-1}$ pattern leads to Σ_g (up to diffeomorphism).

Then the hyperbolic Riemannian metric on D induces a Riemannian metric on Σ_g (because the angles sum up to 2π at the single "vertex" of Σ_g). This Riemannian metric of Σ_g has constant sectional curvature -1. Therefore, the Riemannian universal covering of Σ_g is isometric to $\mathbb{H}^2$ (Theorem 4.4.6). In particular, the universal covering is diffeomorphic to $\mathbb{R}^2$.

In view of Theorem 2.5.11, Riemannian manifolds whose sectional curvature is constant -1 are also called *hyperbolic manifolds*.

4.5 Positive curvature

As a second application of the comparison results, we consider the case of positive curvature, i.e., the case of Riemannian manifolds whose Ricci curvature is uniformly positive. In positive curvature, we expect that the manifold has to "bend" so much that it needs to be of "small" diameter. This is made precise by the Bonnet-Myers theorem.

4.5.1 The Bonnet-Myers theorem

Complete manifolds with uniformly positive Ricci curvature have uniformly bounded diameter; in particular, they have finite fundamental group. This is in stark contrast with the case of non-positive sectional curvature (Outlook 4.4.3).

Theorem 4.5.1 (Bonnet-Myers)**.** *Let (M, g) be a connected complete non-empty Riemannian manifold of dimension $n \geq 2$ and let $R \in \mathbb{R}_{>0}$ with*

$$\forall_{x \in M} \quad \forall_{v \in \mathrm{T}_x M} \quad \mathrm{Ric}_x(v, v) \geq \frac{n-1}{R^2} \cdot \|v\|_g^2.$$

Then, we have:

1. *The diameter of M with respect to d_g is at most $\pi \cdot R$.*
2. *In particular, M is compact.*
3. *The fundamental group of M is finite.*

Proof. Ad 1. Assume for a contradiction that the diameter of (M, d_g) is strictly larger than $\pi \cdot R$. I.e., there exist $x, y \in M$ with $L := d_g(x, y) > \pi \cdot R$. As (M, g) is complete, in view of the Hopf-Rinow theorem (Proposition 3.3.5), there exists a minimising unit speed geodesic $\gamma \colon [0, L] \longrightarrow M$ from x to y.

By the index criterion (Corollary 3.5.5), we then know that all proper normal vector fields V along γ satisfy

$$\mathrm{I}_\gamma(V, V) \geq 0.$$

The idea is now to reach a contradiction by finding a proper normal vector field V along γ with $\mathrm{I}_\gamma(V, V) < 0$.

Let $(E_1, \dots, E_n)$ be a parallel orthonormal frame along γ with $E_1 = \dot{\gamma}$. For $j \in \{2, \dots, n\}$, we consider

$$\begin{aligned} V_j \colon [0, L] &\longrightarrow \mathrm{T}\, M \\ t &\longmapsto \sin\Bigl(\frac{\pi \cdot t}{L}\Bigr) \cdot E_j(t), \end{aligned}$$

which is a proper normal vector field along γ (check!).

We compute

$$\mathrm{I}_\gamma(V_j, V_j) = \int_0^L \sin^2\Bigl(\frac{\pi \cdot t}{L}\Bigr) \cdot \Bigl(\frac{\pi^2}{L^2} - \mathrm{Rm}_{\gamma(t)}\bigl(E_j(t), \dot\gamma(t), \dot\gamma(t), E_j(t)\bigr)\Bigr)\, dt;$$

as usual, this involves compatibility of the Levi-Civita connection with g (Proposition 2.3.7) and straightforward computations (check!). Summing up all contributions leads to

$$\begin{aligned} \sum_{j=2}^n \mathrm{I}_\gamma(V_j, V_j) &= \sum_{j=2}^n \int_0^L \sin^2\Bigl(\frac{\pi \cdot t}{L}\Bigr) \cdot \Bigl(\frac{\pi^2}{L^2} - \mathrm{Rm}_{\gamma(t)}\bigl(E_j(t), \dot\gamma(t), \dot\gamma(t), E_j(t)\bigr)\Bigr)\, dt \\ &= \int_0^L \sin^2\Bigl(\frac{\pi \cdot t}{L}\Bigr) \cdot \Bigl(\frac{(n-1) \cdot \pi^2}{L^2} - \mathrm{Ric}_{\gamma(t)}\bigl(\dot\gamma(t), \dot\gamma(t)\bigr)\Bigr)\, dt && \text{(Proposition 2.4.28)} \\ &\leq \int_0^L \sin^2\Bigl(\frac{\pi \cdot t}{L}\Bigr) \cdot \Bigl(\frac{(n-1) \cdot \pi^2}{L^2} - \frac{n-1}{R^2}\Bigr)\, dt && \text{(curvature hypothesis)} \\ &< 0. && (\text{as } L > \pi \cdot R) \end{aligned}$$

Therefore, at least one of the terms $\mathrm{I}_\gamma(V_j, V_j)$ with $j \in \{2, \dots, n\}$ is negative. This gives the desired contradiction.

Ad 2. Let $x \in M$. As M is complete, we have $\mathrm{Exp}_x = \mathrm{T}_x\, M$. Moreover, by the conjugate point comparison theorem (Theorem 4.3.3) and the first part, we have

$$\exp_x\bigl(\overline{B}^{\mathrm{T}_x M}_{\pi \cdot R}(0)\bigr) = M.$$

Because the closed ball in $\mathrm{T}_x\, M$ (with respect to the radial distance) is compact and $\exp_x$ is continuous, we obtain that M is also compact.

(There are also other ways of deducing the second part from the first part.)

Ad 3. Let $(\widetilde{M}, \widetilde{g})$ be the Riemannian universal covering of (M, g). Then $(\widetilde{M}, \widetilde{g})$ is locally isometric to (M, g). In particular, $(\widetilde{M}, \widetilde{g})$ satisfies the same Ricci curvature bound as (M, g) (Remark 2.4.29). Moreover, $(\widetilde{M}, \widetilde{g})$ is also complete (Exercise).

Therefore, we can apply the second part to $(\widetilde{M}, \widetilde{g})$ and we obtain that $\widetilde{M}$ is compact as well. Therefore, the covering $\widetilde{M} \longrightarrow M$ has only finitely many sheets. As the number of sheets of the universal covering coincides with the cardinality of the fundamental group, we conclude that the fundamental group of M is finite. □

Remark 4.5.2. If (M, g) is a compact Riemannian manifold with sec > 0, then the Bonnet-Myers theorem (Theorem 4.5.1) can be applied as follows: As M

is compact and $\sec > 0$, there exists a $c \in \mathbb{R}_{>0}$ with $\sec \geq c > 0$ (Exercise). Then, we can apply the Bonnet-Myers theorem with $R =$? .

Example 4.5.3. By the Bonnet-Myers theorem (Theorem 4.5.1), the following smooth manifolds do *not* admit a complete Riemannian metric of positive sectional or (uniformly) positive Ricci curvature:

1. $\mathbb{R}^n$ with $n \geq 2$ because
2. $\mathbb{S}^1 \times \mathbb{S}^2$ because
3. the 2-torus and oriented closed connected surfaces of genus at least 2 because

4.5.2 Constant positive sectional curvature

Theorem 4.5.4 (constant positive sectional curvature)**.** *Let* (M, g) *be a complete, simply connected, non-empty Riemannian manifold with constant sectional curvature* $c > 0$ *and let* $n := \dim M \geq 2$*. Then* (M, g) *is isometric to* $\mathbb{S}^n(R)$*, where* $R := 1/\sqrt{c}$*.*

Proof. We argue similarly to the proof of the corresponding result in non-positive curvature (Theorem 4.4.6). However, things are more complicated because we will need more than one normal neighbourhood. We will construct a map $\mathbb{S}^n(R) \longrightarrow M$ that is a Riemannian isometry, using two different normal neighbourhoods.

Let N and S be the north and south pole of $\mathbb{S}^n(R)$, respectively. We first construct local isometries on two normal neighbourhoods of $\mathbb{S}^n$ and then combine these through a glueing argument.

- Let $U := \mathbb{S}^n(R) \setminus \{S\}$. Then the exponential map $\exp_N$ induces a diffeomorphism from the radial ball $B_0^{\mathrm{T}_N \mathbb{S}^n(R)}(\pi \cdot R) \longrightarrow U$ (check!).

  If $x \in M$, then the conjugate point comparison theorem (Theorem 4.3.3) and Proposition 3.5.14 show that $\exp_x \colon B^{\mathrm{T}_x M}\pi \cdot R(0) \longrightarrow M$ is a local diffeomorphism.

  Let $f \colon \mathrm{T}_N \mathbb{S}^n(R) \longrightarrow \mathrm{T}_x M$ be a linear isometry (this exists; check!). Then both
  $$(\exp_x \circ f)^* g \quad \text{and} \quad \exp_N^* g_R$$
  are Riemannian metrics on $B_0^{\mathrm{T}_N \mathbb{S}^n(R)}(\pi \cdot R)$ that both have constant sectional curvature ?

  Moreover, orthonormal coordinates on $\mathrm{T}_N \mathbb{S}^n(R)$ are normal coordinates for both Riemannian metrics (check!). Therefore, these constant sectional curvature metrics have to be equal (Theorem 4.2.4).

In conclusion, the map

$$F := \exp_x \circ f \circ \exp_N^{-1} \colon U \longrightarrow M$$

is a local isometry.

- Let $P \in \mathbb{S}^n(R) \setminus \{N, S\} \subset U$, let $V := \mathbb{S}^n(R) \setminus \{P\}$, and let $y := F(P) \in M$. Then the linear isometry $\widetilde{f} := \mathrm{d}_P f \colon \mathrm{T}_P \mathbb{S}^n(R) \longrightarrow \mathrm{T}_y M$ similarly leads to a local isometry

$$\widetilde{F} := \exp_y \circ \widetilde{f} \circ \exp_P^{-1} \colon V \longrightarrow M.$$

- By construction,

$$F(P) = \widetilde{F}(P) \quad \text{and} \quad \mathrm{d}_P F = \mathrm{d}_P \widetilde{F}$$

(check!). Therefore, on $U \cap V$, these local isometries have to coincide (this follows as in Proposition 3.1.15).

Therefore, F and $\widetilde{F}$ glue to a local isometry

$$\varphi \colon (\mathbb{S}^n(R), g_R) \longrightarrow (M, g).$$

- For the final step, we make use of topology: The local isometry φ is a covering map (Exercise). Because $\mathbb{S}^n(R)$ and M both are simply connected, covering theory implies that φ has to be a homeomorphism. As we already know that φ is a local isometry, it follows that φ is an isometry (check!). □

Study note. Can you still remember how (vastly more general) theorems of this type enter in the proof of the Poincaré conjecture?

Example 4.5.5 (non-existence of Riemannian metrics of constant sectional curvature)**.** For both of the following examples let us recall that Riemannian metrics on compact manifolds are always complete (Example 3.3.3).

- The manifold $\mathbb{S}^2 \times \mathbb{S}^2$ does *not* admit a Riemannian metric of constant sectional curvature:

  Because $\mathbb{S}^2 \times \mathbb{S}^2$ is simply connected and compact, $\mathbb{S}^2 \times \mathbb{S}^2$ does *not* admit a Riemannian metric of constant sectional curvature ≤ 0 (Outlook 4.4.3).

  Moreover, $\mathbb{S}^2 \times \mathbb{S}^2$ is *not* homeomorphic to $\mathbb{S}^4$ (e.g., one can use the second homology or the second homotopy group). Therefore, $\mathbb{S}^2 \times \mathbb{S}^2$ does *not* admit a Riemannian metric of constant positive sectional curvature (Theorem 4.5.4).

- If M is a compact manifold with fundamental group isomorphic to $\mathbb{Z}/2\times \mathbb{Z}/2$, then M does *not* admit a Riemannian metric of constant sectional curvature:

  Because $\mathbb{Z}/2 \times \mathbb{Z}/2$ is finite and M is compact, M does *not* admit a Riemannian metric of constant sectional curvature ≤ 0 (Outlook 4.4.3).

  Assume for a contradiction that M admits a Riemannian metric of constant positive sectional curvature. Then the universal covering of M is homeomorphic to $\mathbb{S}^{\dim M}$ (Theorem 4.5.4) and the fundamental group $\mathbb{Z}/2 \times \mathbb{Z}/2$ acts by deck transformations on this sphere. Moreover, this action is free (by covering theory). However, this contradicts the fact that $\mathbb{Z}/2 \times \mathbb{Z}/2$ does *not* have periodic group cohomology and finite groups acting freely on spheres must have periodic group cohomology [20, Chapter 4.3].

4.6 The Švarc-Milnor lemma

Finally, we briefly outline a beautiful connection between Riemannian geometry, algebraic topology, and geometric group theory: The Švarc-Milnor lemma.

The Švarc-Milnor lemma relates the geometry of fundamental groups to compact Riemannian manifolds to the geometry of the Riemannian universal covering. In particular, this will relate their volume growth behaviour. We will only formulate the most basic version of this result.

For this, we first introduce word metrics and growth rates on finitely generated groups. We then define a suitable notion of geometric equivalence (quasi-isometry) and state the Švarc-Milnor theorem and some of its consequences.

Definition 4.6.1 (word metric and growth functions of groups)**.** Let Γ be a finitely generated group, finitely generated by S.

- The *word metric of* Γ *with respect to* S is defined as

$$\begin{aligned} d_{\Gamma,S} \colon \Gamma \times \Gamma &\longrightarrow \mathbb{N} \\ (g,h) &\longmapsto \min\bigl\{n \in \mathbb{N} \bigm| \exists_{s_1,\dots,s_n \in S \cup S^{-1}} \quad g^{-1} \cdot h = s_1 \cdot \dots \cdot s_n\bigr\}. \end{aligned}$$

- The *growth function of* Γ *with respect to* S is

$$\begin{aligned} \beta^{\Gamma,S} \colon \mathbb{N} &\longrightarrow \mathbb{N} \\ r &\longmapsto \bigl|\{g \in \Gamma \mid d_S(g,e) \leq r\bigr|. \end{aligned}$$

Remark 4.6.2 (quasi-geometry)**.** If Γ is a finitely generated group and $S, T \subset \Gamma$ are finite generating sets, then the word metrics $d_{\Gamma,S}$ and $d_{\Gamma,T}$, in general, will not be equal. Similarly, we also cannot expect that the growth functions $\beta^{\Gamma,S}$ and $\beta^{\Gamma,T}$ will be equal. However, these word metrics and growth functions are "quasi-equal" – and this observation is one of the starting points of geometric group theory [8, 19].

More precisely: The identity map id_Γ is a quasi-isometry between $(\Gamma, d_{\Gamma,S})$ and $(\Gamma, d_{\Gamma,T})$. Moreover, $\beta^{\Gamma,S} \sim \beta^{\Gamma,T}$ [8, 19]. These notions are defined as follows:

Let (X, d) and (X', d') be metric spaces.

- A map $f\colon (X,d) \longrightarrow (X',d')$ is a *quasi-isometric embedding* if there exists a $c \in \mathbb{R}_{>0}$ with
$$\forall_{x,y\in X} \quad \frac{1}{c} \cdot d(x,y) - c \leq d'\bigl(f(x), f(y)\bigr) \leq c \cdot d(x,y) + c.$$
- A quasi-isometric embedding $f\colon (X,d) \longrightarrow (X',d')$ is a *quasi-isometry* if there exists a quasi-isometric embedding $g\colon (X',d') \longrightarrow (X,d)$ with
$$\sup_{x\in X} d\bigl(g \circ f(x), x\bigr) < \infty \quad \text{and} \quad \sup_{x\in X'} d'\bigl(f \circ g(x), x\bigr) < \infty.$$

Let $f, g\colon \mathbb{N} \longrightarrow \mathbb{R}$ be (monotonically increasing) functions.

- Then g *quasi-dominates* f, in symbols $f \prec g$, if there exists a $c \in \mathbb{R}_{>0}$ with
$$\forall_{r\in\mathbb{N}} \quad f(r) \leq c \cdot g(c \cdot r + c) + c.$$
- The functions f and g are *quasi-equivalent*, in symbols $f \sim g$, if
$$f \prec g \quad \text{and} \quad g \prec f.$$

Remark 4.6.3 (maximal growth of groups)**.** If Γ is a finitely generated group, finitely generated by S, then
$$\forall_{r\in\mathbb{N}} \quad \beta^{\Gamma,S}(r) \leq \bigl(2 \cdot |S| + 1\bigr)^r.$$
Therefore, finitely generated groups have at most exponential growth.

Example 4.6.4 (growth of groups)**.**

- The growth function of $\mathbb{Z}^2$ with respect to the standard basis is (roughly) quadratic (check!).
- The growth function of the free group of rank 2 with respect to a free generating set is (roughly) exponential (check!).

Theorem 4.6.5 (Švarc-Milnor lemma). *Let (M, g) be a compact Riemannian manifold and let $x \in M$. Moreover, let $\pi \colon (\widetilde{M}, \widetilde{g}) \longrightarrow (M, g)$ be the Riemannian universal covering of (M, g) and let $\widetilde{x} \in \widetilde{M}$ be a π-lift of x.*

1. *Then the fundamental group $\Gamma := \pi_1(M, x)$ of M is finitely generated by*
$$S := \{s \in \Gamma \mid s \cdot D \cap D \neq \emptyset\}$$
where R denotes the (finite!) diameter of (M, d_g) and D is the $4 \cdot R$-neighbourhood of $B^{\widetilde{M}, d_{\widetilde{g}}}_{2 \cdot R}(\widetilde{x})$.
2. *The map*
$$\begin{aligned} f \colon \Gamma &\longrightarrow \widetilde{M} \\ \gamma &\longmapsto \gamma \cdot x \end{aligned}$$
is a quasi-isometry with respect to the word metric $d_{\Gamma,S}$ on Γ and $d_{\widetilde{g}}$ on $\widetilde{M}$.
3. *The growth functions $\beta^{\Gamma,S}$ and $\varrho^{(\widetilde{M},\widetilde{g})}_{\widetilde{x}}$ are quasi-equivalent.*

The proof relies on the connect-the-dots principle and requires only elementary metric geometry, Riemannian geometry, and basic covering theory [23, 8, 5, 19]. It is highly recommend to study this proof!

Corollary 4.6.6. *Let (M, g) be a compact non-empty Riemannian manifold.*

1. *Then the volume growth of the Riemannian universal covering $(\widetilde{M}, \widetilde{g})$ is at most exponential: For each $\widetilde{x} \in \widetilde{M}$, there exists an $a \in \mathbb{R}_{\geq 1}$ such that*
$$\forall_{r \in \mathbb{R}_{>0}} \quad \varrho^{\widetilde{M},\widetilde{g}}_{\widetilde{x}}(r) \leq a^r.$$
2. *If $\sec^{M,g} < 0$, then the fundamental group of M has exponential growth.*

Proof. Ad 1. By Remark 4.6.3, the fundamental group $\pi_1(M)$ has at most exponential growth. Therefore, by the Švarc-Milnor lemma (Theorem 4.6.5), the same holds for the volume growth of $(\widetilde{M}, \widetilde{g})$.

Ad 2. Because M is compact and $\sec < 0$, there exists a constant $c \in \mathbb{R}_{<0}$ with $\sec^{M,g} \leq c < 0$ (Exercise). Therefore, the Riemannian universal covering $(\widetilde{M}, \widetilde{g})$ satisfies $\sec^{\widetilde{M},\widetilde{g}} \leq c < 0$ (Remark 2.4.21). In particular, $(\widetilde{M}, \widetilde{g})$ has at least exponential growth (Corollary 4.4.5).

Thus, by the first part and the Švarc-Milnor lemma, the fundamental group $\pi_1(M)$ has exponential growth. $\square$

Example 4.6.7. The 2-torus $\mathbb{S}^1 \times \mathbb{S}^1$ does *not* admit a Riemannian metric of negative sectional curvature: The fundamental group of $\mathbb{S}^1 \times \mathbb{S}^1$ is isomorphic to $\mathbb{Z}^2$ and thus has polynomial growth (Example 4.6.4). Now Corollary 4.6.6 finishes the argument.

Outlook 4.6.8 (polynomial growth). By *Gromov's polynomial growth theorem* [11, 8, 19], a finitely generated group has polynomial growth if and only if it contains a nilpotent subgroup of finite index. Moreover, Gromov's original application is geometric [11].

For example, in combination with the Švarc-Milnor theorem, this implies that there is *no* compact Riemannian manifold whose Riemannian universal covering has polynomial volume growth of non-integral degree [19].

Study note (summary). Write a summary of Chapter 4, keeping the following questions in mind:

- What are comparison theorems and which tools did we use to derive them?
- How can manifolds of constant sectional curvature be characterised?
- What can be said about the global shape of manifolds of non-positive sectional curvature?
- What can be said about the global shape of manifolds of positive sectional curvature?
- How can one exclude the existence of certain types of Riemannian manifolds?

Study note. This is not the end, but only the beginning: There are lots of directions to explore, for instance, comparison geometry, geometric group theory, geometric analysis, and mathematical physics. Enjoy!

A

Appendix

Overview of this chapter.

A.1 Categories and functors

Mathematical theories consist of objects (e.g., groups, topological spaces, ...) and structure preserving maps (e.g., group homomorphisms, continuous maps, ...). This can be abstracted to the notion of a category.

A.1.1 Categories

Definition A.1.1 (category). A *category* C consists of the following data:

- A class $\mathrm{Ob}(C)$; the elements of $\mathrm{Ob}(C)$ are called *objects of* C.
- For all objects $X, Y \in \mathrm{Ob}(C)$ a set $\mathrm{Mor}_C(X, Y)$; the elements of the set $\mathrm{Mor}_C(X, Y)$ are called *morphisms from* X *to* Y *in* C. (Implicitly, we will assume that the morphism sets between different pairs of objects are disjoint and that we can recover the source and target object from a morphism.)
- For all objects $X, Y, Z \in \mathrm{Ob}(C)$ a composition

$$\begin{aligned} \circ\colon \mathrm{Mor}_C(Y, Z) \times \mathrm{Mor}_C(X, Y) &\longrightarrow \mathrm{Mor}_C(X, Z) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

  of morphisms.

This data is required to satisfy the following conditions:

- For each object X in C there exists a morphism $\mathrm{id}_X \in \mathrm{Mor}_C(X, X)$ such that: For all $Y \in \mathrm{Ob}(C)$ and all morphisms $f \in \mathrm{Mor}_C(X, Y)$ and $g \in \mathrm{Mor}_C(Y, X)$, we have

$$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_X \circ g = g.$$

  (The morphism id_X is uniquely determined by this property (check!); it is the *identity morphism of* X *in* C.)

- The composition of morphism is associative: For all objects W, X, Y, Z in C and all morphisms $f \in \mathrm{Mor}_C(W, X)$, $g \in \mathrm{Mor}_C(X, Y)$, and $h \in \mathrm{Mor}_C(Y, Z)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Remark A.1.2 (classes). Classes are a tool to escape the set-theoretic paradoxon of the "set of all sets" (Chapter I.1.3.3). In case you are not familiar with von Neumann-Bernays-Gödel set theory, you can use the slogan that classes are "potentially large", "generalised" setss.

All concepts and facts in mathematical theories that can be expressed in terms of objects, identity morphisms, and (the composition of) morphisms also admit a category theoretic version. For instance, in this way, we obtain a general notion of isomorphism:

Definition A.1.3 (isomorphism)**.** Let C be a category. Objects $X, Y \in \operatorname{Ob}(C)$ are *isomorphic in* C, if there exist morphisms $f \in \operatorname{Mor}_C(X, Y)$ and $g \in \operatorname{Mor}_C(Y, X)$ with

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y .$$

In this case, f and g are *isomorphisms in* C and we write $X \cong_C Y$. If the category is clear from the context, we might also write $X \cong Y$.

Proposition A.1.4 (elementary properties of isomorphisms)**.** *Let C be a category and let $X, Y, Z \in \operatorname{Ob}(C)$.*

1. *Then the identity morphism id_X is an isomorphism in C (from X to X).*
2. *If $f \in \operatorname{Mor}_C(X, Y)$ is an isomorphism in C, then there is a unique morphism $g \in \operatorname{Mor}_C(Y, X)$ that satisfies $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.*
3. *Compositions of (composable) isomorphisms are isomorphisms.*
4. *If $X \cong_C Y$, then $Y \cong_C X$.*
5. *If $X \cong_C Y$ and $Y \cong_C Z$, then $X \cong_C Z$.*

Proof. All claims follow easily follow from the definitions (check!). $\square$

Example A.1.5 (set theory)**.** The category Set of sets consists of:

- objects: Let $\operatorname{Ob}(\mathsf{Set})$ be the class(!) of all sets.
- morphisms: If X and Y are sets, then we define $\operatorname{Mor}_{\mathsf{Set}}(X, Y)$ as the set of all set-theoretic maps $X \longrightarrow Y$.
- compositions: If X, Y, and Z are sets, then the composition map $\operatorname{Mor}_{\mathsf{Set}}(Y, Z) \times \operatorname{Mor}_{\mathsf{Set}}(X, Y) \longrightarrow \operatorname{Mor}_{\mathsf{Set}}(X, Z)$ is ordinary composition of maps.

Clearly, this composition is associative. If X is a set, then the usual identity map

$$\begin{aligned} X &\longrightarrow X \\ x &\longmapsto x \end{aligned}$$

is the identity morphism id_X of X in Set. Objects in Set are isomorphic if and only if there exists a bijection between them, i.e., if they have the same cardinality.

Caveat A.1.6. The concept of morphisms and compositions in the definition of categories is modelled on the example of maps between sets and ordinary composition of maps. In general categories, morphisms are not necessarily maps between sets and the composition of morphisms is not necessarily ordinary composition of maps!

Example A.1.7 (algebra)**.** Let K be a field. The category Vect_K of K-vector spaces consists of:

- objects: Let $\mathrm{Ob}(\mathsf{Vect}_K)$ be the class(!) of all K-vector spaces.
- morphisms: If X, Y are K-vector spaces, then we define $\mathrm{Mor}_{\mathsf{Vect}_K}(X, Y)$ as the set of all K-linear maps $X \longrightarrow Y$. In this case, we also write $\mathrm{Hom}_K(X, Y)$ for the set of morphisms.
- compositions: As composition we take the ordinary composition of maps.

Objects in Vect_K are isomorphic if and only if they are isomorphic in the classical sense from Linear Algebra.

Analogously, we can define the category Group of groups, the category Ab of Abelian groups, the category ${}_R\mathsf{Mod}$ of left-modules over a ring R, the category Mod_R of right-modules over a ring R, ...

A.1.2 Functors

As next step, we will formalise translations between mathematical theories, using functors. Roughly speaking, functors are "structure preserving maps between categories" (Figure A.1).

Definition A.1.8 (functor)**.** Let C and D be categories. A *(covariant) functor* $F\colon C \longrightarrow D$ consists of the following data:

- A map $F\colon \mathrm{Ob}(C) \longrightarrow \mathrm{Ob}(D)$.
- For all objects $X, Y \in \mathrm{Ob}(C)$ a map

$$F\colon \mathrm{Mor}_C(X, Y) \longrightarrow \mathrm{Mor}_D\bigl(F(X), F(Y)\bigr).$$

This data is required to satisfy the following conditions:

- For all $X \in \mathrm{Ob}(C)$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
- For all $X, Y, Z \in \mathrm{Ob}(C)$ and all $f \in \mathrm{Mor}_C(X, Y)$, $g \in \mathrm{Mor}_C(Y, Z)$, we have

$$F(g \circ f) = F(g) \circ F(f).$$

A *contravariant functor* $F\colon C \longrightarrow D$ consists of the following data:

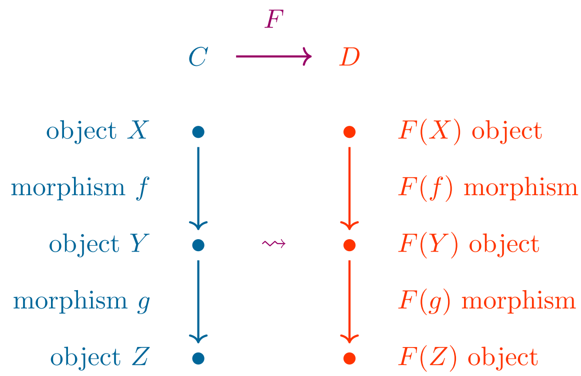


Figure A.1.: Functor, schematically

- A map $F\colon \operatorname{Ob}(C) \longrightarrow \operatorname{Ob}(D)$.
- For all objects $X, Y \in \operatorname{Ob}(C)$ a map

$$F\colon \operatorname{Mor}_C(X,Y) \longrightarrow \operatorname{Mor}_D\bigl(F(Y), F(X)\bigr).$$

This data is required to satisfy the following conditions:

- For all $X \in \operatorname{Ob}(C)$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
- For all $X, Y, Z \in \operatorname{Ob}(C)$ and all $f \in \operatorname{Mor}_C(X,Y)$, $g \in \operatorname{Mor}_C(Y,Z)$, we have

$$F(g \circ f) = F(f) \circ F(g).$$

In other words, contravariant functors reverse the direction of arrows. More concisely, contravariant functors $C \longrightarrow D$ are the same as covariant functors $C \longrightarrow D^{\mathrm{op}}$, where D^{op} denotes the dual category of D.

The key property of functors is that they preserve isomorphisms. In particular, functors provide a good notion of invariants.

Proposition A.1.9 (functors preserve isomorphism)**.** *Let C and D be categories, let $F\colon C \longrightarrow D$ be a functor, and let $X, Y \in \operatorname{Ob}(C)$.*

1. *If $f \in \operatorname{Mor}_C(X,Y)$ is an isomorphism in C, then the translated morphism $F(f) \in \operatorname{Mor}_D(F(X), F(Y))$ is an isomorphism in D.*
2. *In particular: If $X \cong_C Y$, then $F(X) \cong_D F(Y)$. In other words: If $F(X) \not\cong_D F(Y)$, then $X \not\cong_C Y$.*

Proof. The first part follows from the defining properties of functors: Because f is an isomorphism, there is a morphism $g \in \operatorname{Mor}_C(Y,X)$ with

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y .$$

Hence, we obtain

$$F(g) \circ F(f) = F(g \circ f) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$$

and $F(f) \circ F(g) = \mathrm{id}_{F(Y)}$. Thus, $F(f)$ is an isomorphism from $F(X)$ to $F(Y)$ in D.

The second part is a direct consequence of the first part. □

Therefore, suitable functors can help to prove that certain objects are *not* isomorphic.

Caveat A.1.10. In general, the converse is *not* true! I.e., objects that are mapped via a functor to isomorphic objects are, in general, *not* isomorphic.

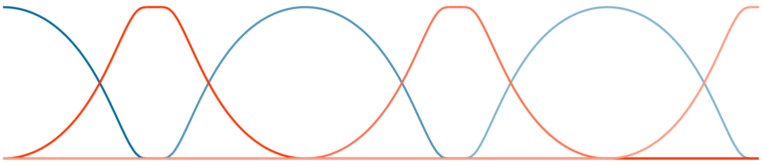
A.2 Partitions of unity

Partitions of unity allow to glue convex local data on manifolds to global objects. Roughly speaking, a partition of unity is a family of real-valued functions with values in $[0,1]$ that sum at every point to 1 (whence the name; Figure A.2).

Theorem A.2.1 (existence of partitions of unity [1, Proposition 2.22, Korollar 2.23])**.** *Let M be a smooth manifold and let $(U_i)_{i\in I}$ be an open cover of M. Then M admits a* partition of unity that refines *$(U_i)_{i\in I}$, i.e., there exists a family $(\varphi_i)_{i\in I}$ in $C^\infty(M)$ with the following properties:*

1. *For all $i \in I$ and $x \in M$, we have $0 \le \varphi_i(x) \le 1$.*
2. *For all $i \in I$, we have $\operatorname{supp}\varphi_i \subset U_i$.*
3. *The family $(\operatorname{supp}\varphi_i)_{i\in I}$ is a locally finite cover of M.*
4. *For all $x \in M$, we have*
$$\sum_{i\in I} \varphi_i(x) = 1.$$

The proof is based on smooth bump functions and the statement can be refined further, e.g., to partition functions with compact support.



Toggle function 1 Toggle function 4

Toggle function 2 Toggle function 5

Toggle function 3 Toggle function 6

Figure A.2.: An example of a partition of unity on $\mathbb{R}$

A.3 Group actions

We briefly recall basic terminology of group actions (of discrete groups). Group actions are a generalisation of the notion of symmetry. This general concept can be nicely formalised in the language of categories:

Definition A.3.1 (group action)**.** Let C be a category, let X be an object in C, and let G be a group.

- The *automorphism group of X in C* is the group(!) $\operatorname{Aut}_C(X)$ (with respect to composition in C) of all isomorphisms from X to X in C.
- A *group action of G on X in C* is a group homomorphism

$$G \longrightarrow \operatorname{Aut}_C(X).$$

  We also (sloppily) denote such an action by $G \curvearrowright X$.
- A *right action of G on X in C* is a group anti-homomorphism

$$\varphi\colon G \longrightarrow \operatorname{Aut}_C(X),$$

  i.e., for all $g, h \in G$ we have $\varphi(g \cdot h) = \varphi(h) \circ \varphi(g)$. We also denote this by $X \curvearrowleft G$.

Group actions can often be described in more explicit terms:

Remark A.3.2 (group actions in Set, Top, …)**.** Let G be a group and let X be a set [a topological space]. A map $\varphi\colon G \curvearrowright X$ is a group action of G on X in Set [in Top] if and only if the map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g \cdot x := \bigl(\varphi(g)\bigr)(x) \end{aligned}$$

has the following properties:

- For every $g \in G$, the map $g \cdot \;\cdot\; \colon X \longrightarrow X$ is a map of sets [a continuous map]
- For all $x \in X$, we have $e \cdot x = x$.
- For all $x \in X$ and all $g, h \in G$, we have

$$(g \cdot h) \cdot x = g \cdot (h \cdot x).$$

[Group actions in Top are also called *continuous actions*.]

Similarly, *smooth actions* are group actions on smooth manifolds by smooth maps; *isometric actions* are group actions on metric spaces by isometries.

Elementary examples of group actions have been covered in the Algebra course (Chapter III.1.2).

Definition A.3.3 (free action, stabiliser, orbit, orbit space, transitive)**.** Let $G \curvearrowright X$ be a group action of G on X in Set [in Top].

- The action is *free*, if the following holds: For all $x \in X$ and all $g \in G \setminus \{e\}$, we have $g \cdot x \neq x$.
- Let $x \in X$. The *stabiliser* of the action at x is the subgroup
$$G_x := \{g \in G \mid g \cdot x = x\} \subset G.$$
- For $x \in X$, we write
$$G \cdot x := \{g \cdot x \mid g \in G\} \subset X$$
for the *orbit of* x.
- The quotient space
$$G \setminus X := \{G \cdot x \mid x \in X\}$$
is the *orbit space* of this action. . [For actions in Top, we endow the orbit space $G \setminus X$ with the quotient topology induced by the canonical projection $X \longrightarrow G \setminus X$.]
- The action is *transitive*, if $|G \setminus X| = 1$, i.e., if all points in X lie in the same orbit.

Analogously, we introduce the corresponding terms for right actions. The orbit space of a right action $X \curvearrowleft G$ is denoted by X/G.

Caveat A.3.4. The quotient topology on the orbit space of a continuous action can be terrible even if the group and the space acted upon are "nice". In order to have a "nice" quotient space, the *action* needs to have good properties.

B

Assignments and exercise sheets

Differential Geometry I: Week 1

Prof. Dr. C. Löh/AG Ammann November 4, 2020

Reading assignment (for the lecture on November 4).

- Register on GRIPS for this course.

In the first lecture, we will get acquainted with the video conferencing tool, we will discuss organisational matters, and I will give a brief overview of this course.

After this first lecture, the reading material for the second meeting will be made available. In subsequent weeks with remote teaching, I will always try to provide the material for the whole week $n + 1$ on the Thursday of week n.

Reading assignment (for the lecture on November 5).

- Think about how you can efficiently and successfully organise self-study: for example, when to work on the reading assignments, how to make sure that you do not just browse quickly through the material but get immersed into the subject and spend enough time on it, when/how to "meet" with fellow students to discuss the topics, which notes/annotations to take during interactive sessions, how to organise questions that arise while reading (and when/where to ask them), which additional sources to consult, how to integrate working on the exercise sheets, how to use your hardware setup in an effective way, how to schedule breaks from working at the computer screen, ...
- Register for the text chat and start interacting with other participants!
- Read Chapter 1.1.2 *Smooth manifolds.*
- Read Chapter 1.1.3 *The category of smooth manifolds.*
- Read Chapter 1.1.4 *Tangent spaces.*

This is not a lot of new material, but you will only have one day to prepare ... From week 2 on, there will be more Mathematics!

Étude (differentials). Let M be a smooth manifold and $x \in M$. Compute $\mathrm{d}_x f$ for the following choices of f:

1. a constant map $M \longrightarrow M$
2. the projection $M \times M \longrightarrow M$ onto the first factor
3. the diagonal map $M \longrightarrow M \times M$
4. the inclusion $M \longrightarrow M \times \mathbb{R}$ into $M \times \{0\}$

Hints. Solutions to *Études* are not be submitted and will not be graded.

Exercises (for the session on November 9/10). In this first exercise session, some basics on smooth manifolds will be discussed (e.g., as in the following exercises).

Please turn over

Exercise 0.1 (compact manifolds)**.** Let M be a smooth manifold. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If M is compact, then M admits a finite smooth atlas.
2. If M admits a finite smooth atlas, then M is compact.

Exercise 0.2 (smooth atlas on $\mathbb{R}$)**.** We equip $\mathbb{R}$ with the standard topology and consider the map

$$\begin{aligned} \varphi \colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} x & \text{if } x \leq 0, \\ 2 \cdot x & \text{if } x > 0. \end{cases} \end{aligned}$$

Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. The set $\{\varphi\}$ is a smooth atlas for $\mathbb{R}$.
2. The set $\{\varphi, \mathrm{id}_{\mathbb{R}}\}$ is a smooth atlas for $\mathbb{R}$.

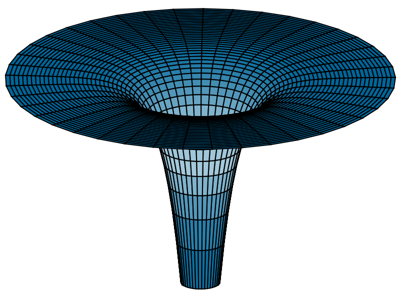
Exercise 0.3 (local coordinates)**.** Let $\alpha \in \mathbb{R}$ and let $f \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation around 0 about the angle α.

1. Express f in local coordinates with respect to the smooth chart $\mathrm{id}_{\mathbb{R}^2}$.
2. Express f in local coordinates with respect to polar coordinates.

Exercise 0.4 (differentials via derivations)**.** Let $f \colon M \longrightarrow N$ be a smooth map between smooth manifolds and let $x \in M$.

1. Describe the differential $\mathrm{d}_x f \colon \mathrm{T}_x M \longrightarrow \mathrm{T}_x N$ in terms of derivations.
2. Prove that this description coincides with the description of $\mathrm{d}_x f$ in terms of curves.

Exercise 0.5 (the real world)**.** Give examples of "real-world" situations that can be modelled by smooth manifolds.



No submission! Questions of this type will be discussed in the first exercise session on November 9/10.

Differential Geometry I: Week 2

Prof. Dr. C. Löh/AG Ammann November 5, 2020

Reading assignment (for the lecture on November 11). We finish our quick recap of smooth manifolds by recalling submanifolds. After that, we will introduce the language of (smooth) vector bundles.

- Read Chapter 1.1.5 *Submanifolds.*
- Read Chapter 1.2.1 *Smooth vector bundles.*
- Read Chapter 1.2.2 *Constructing vector bundles* until Proposition 1.2.7.

Reading assignment (for the lecture on November 12).

- Read the rest of Chapter 1.2.2 *Constructing vector bundles.*
- Read Chapter 1.2.3 *The tangent bundle.*
- Read Appendix A.1 *Categories and functors.* We will not use categories and functors in a serious way, but it is convenient to be able to use this language now and then.

Next week, we will construct tensor bundles of smooth manifolds and finally we will introduce Riemannian metrics.

Étude (regular values). Which of the following values are regular values of the given maps?

1. 2020 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $x \longmapsto \|x\|_2{}^2$
2. 2020 of $M_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}$, $A \longmapsto \operatorname{tr} A$
3. 2020 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto y^2 - x^3$
4. 0 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto y^2 - x^3$
5. 2020 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto x \cdot y$
6. 0 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto x \cdot y$

Exercises (for the session on November 16/17). The following exercises (which all are solvable with the material read/discussed in week 1) will be discussed.

Please turn over

Exercise 1.1 (smooth charts)**.** Let M be a smooth manifold of dimension n and let $U \longrightarrow U'$ and $V \longrightarrow V'$ be smooth charts of M. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. We have $U' \cap V' \cong_{\mathsf{Top}} U \cap V$.
2. If $U \cong_{\mathsf{Top}} \mathbb{R}^n$ and $V \cong_{\mathsf{Top}} \mathbb{R}^n$, then $U \cap V = \emptyset$ or $U \cap V \cong_{\mathsf{Top}} \mathbb{R}^n$.

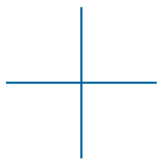
Exercise 1.2 (local vs. global diffeomorphisms)**.** A smooth map $f \colon M \longrightarrow N$ between smooth manifolds is a *local diffeomorphism* if for each $x \in M$, there exists an open neighbourhood $U \subset M$ of x in M such that $f(U) \subset N$ is open and $f|_U \colon U \longrightarrow f(U)$ is a diffeomorphism.

1. Show that not every surjective local diffeomorphism is a diffeomorphism.
2. Show that bijective local diffeomorphisms are diffeomorphisms.

Exercise 1.3 (a non-manifold)**.** Show that the set

$$\bigl\{(x, y) \in \mathbb{R}^2 \bigm| x \cdot y = 0\bigr\} \subset \mathbb{R}^2$$

is *not* a topological manifold with respect to the subspace topology inherited from the standard topology on $\mathbb{R}^2$. Illustrate your arguments with suitable pictures!



Exercise 1.4 (smooth currying)**.** Let V and W be finite-dimensional $\mathbb{R}$-vector spaces, let M be a smooth manifold, and let $f \colon M \times V \longrightarrow W$ be a smooth map with the property that for each $x \in M$, the map $f(x, \,\cdot\,) \colon V \longrightarrow W$ is $\mathbb{R}$-linear. Show that then also the "curried" function

$$\begin{aligned} M &\longrightarrow \operatorname{Hom}_{\mathbb{R}}(V, W) \\ x &\longmapsto f(x, \,\cdot\,) \end{aligned}$$

is smooth.
Hints. It might be helpful to think about this in terms of bases.

Bonus problem (the category of smooth manifolds)**.** Justify your answers with suitable arguments!

1. Does the category Mfd of smooth manifolds have an initial object?
2. Does the category Mfd of smooth manifolds have a terminal object?
3. Does the category Mfd of smooth manifolds contain all finite products?
4. Does the categoyr Mfd of smooth manifolds contain all finite coproducts?

Submission before November 12, 2020, 10:00, via email to your tutor.

Differential Geometry I: Week 3

Prof. Dr. C. Löh/AG Ammann November 12, 2020

Reading assignment (for the lecture on November 18). Applying constructions from multilinear algebra to the tangent bundle leads to the tensor bundles of the tangent bundle, which allow to define Riemannian metrics.

- Recall dual spaces, tensor products, and exterior products, including universal properties, constructions, and how to find bases.
- Read Chapter 1.2.4 *Tensor bundles.*
- Read Chapter 1.3.1 *Riemannian metrics* until Quick check 1.3.2.

Reading assignment (for the lecture on November 19).

- Read Appendix A.2 *Partitions of unity* to recall/learn this tool.
- Read the rest of Chapter 1.3.1 *Riemannian metrics.*
- Read Chapter 1.3.2 *Riemannian manifolds.*

Next week, we will spend some time with the three model spaces of Riemannian geometry and start with the study of Riemannian *geometry.*

Étude (Riemannian metrics). Which of the following terms define Riemannian metrics on $\mathbb{R}^2$?

1. $\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^2$
2. $\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1$
3. $\sin(x^1 + x^2) \cdot \mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1 + \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^2$
4. $\frac{1}{e^{x^1} + x^1 \cdot x^1} \cdot (\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1 + \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^2)$
5. $\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1 - \mathrm{d}\,x^1 \otimes \mathrm{d}\,x^2 - \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^1 + 2 \cdot \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^2$
6. $\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1 - \mathrm{d}\,x^1 \otimes \mathrm{d}\,x^2 - \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^1 + \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^2$

Exercises (for the session on November 23/24). The following exercises (which all are solvable with the material read/discussed in week 2) will be discussed.

Please turn over

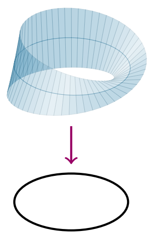
Exercise 2.1 (isomorphic vector bundles)**.** Let M be a smooth manifold and let $\pi\colon E \longrightarrow M$ and $\pi'\colon E' \longrightarrow M$ be smooth vector bundles over M. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If π and π' are isomorphic smooth vector bundles, then $E_x \cong_{\mathbb{R}} E'_x$ for all $x \in M$.
2. If $E_x \cong_{\mathbb{R}} E'_x$ holds for all $x \in M$, then π and π' are isomorphic smooth vector bundles.

Exercise 2.2 (trivial vector bundles)**.** Let $\pi\colon E \longrightarrow M$ be a smooth vector bundle of rank $k \in \mathbb{N}$. Show that the following are equivalent:

- The vector bundle π is trivial.
- The vector bundle admits k linearly independent sections, i.e., there exist sections $s_1, \dots, s_k \in \Gamma(\pi)$ with the property that for all $x \in M$, the family $(s_1(x), \dots, s_k(x))$ in E_x is linearly independent over $\mathbb{R}$.

Exercise 2.3 (Möbius strip)**.** We can view the open Möbius strip as a line bundle over the circle $\mathbb{S}^1$ (see below). Provide a rigorous construction of this line bundle via a suitable cocycle (two patches suffice). Illustrate your arguments with suitable pictures!
Hints. It is instructive to build a paper Möbius strip and to work out bundle theory on this model.

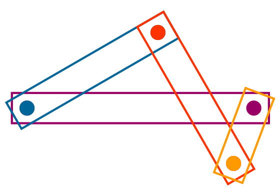


Exercise 2.4 (Möbius strip, embedded)**.** Specify a smooth submanifold of $\mathbb{R}^3$ concretely that is diffeomorphic to the Möbius strip (and prove this fact).
Hints. You can easily visually check whether your subset of $\mathbb{R}^3$ is a valid candidate by plotting it with a suitable program!

Bonus problem (mechanical linkages)**.** We consider the following article:

> M. Kapovich, J. Millson. Universality theorems for configuration spaces of planar linkages, *Topology*, 41(6), pp. 1051–1107, 2002.

1. How can compact smooth manifolds be represented in terms of mechanical linkages? It suffices to briefly introduce the main notions and to cite appropriate results from the article!
2. Does every mechanical linkage lead to a compact smooth manifold?



Submission before November 19, 2020, 10:00, via email to your tutor.

Differential Geometry I: Week 4

Prof. Dr. C. Löh/AG Ammann November 19, 2020

Reading assignment (for the lecture on November 25). We will introduce important families of "symmetric" examples of Riemannian manifolds: spheres, Euclidean spaces, hyperbolic spaces.

- Recall the notion of *group actions* (Appendix A.3).
- Read Chapter *1.4.1 Homogeneous spaces*.
- Read Chapter *1.4.2 Euclidean spaces*.
- Read Chapter *1.4.3 Spheres*.

Reading assignment (for the lecture on November 26).

- Read Chapter *1.4.4 Hyperbolic spaces*.
- (Optional) Get used to the hyperbolic plane by playing *HyperRogue*: https://github.com/zenorogue/hyperrogue

It is time to turn Riemannian geometry into actual geometry: Next week, we will outline the construction of geometric invariants via Riemannian metrics.

Étude (Riemannian isometries). Let g° denote the round Riemannian metric on $\mathbb{S}^1$ and let g be the Euclidean Riemannian metric on $\mathbb{R}^2$. Which of the following maps are local isometries? Isometries?

1. The inclusion $(\mathbb{S}^1, g^\circ) \longrightarrow (\mathbb{R}^2, g)$.
2. The doubling map
$$\begin{aligned}(\mathbb{S}^1, g^\circ) &\longrightarrow (\mathbb{S}^1, g^\circ)\\ (\cos t, \sin t) &\longmapsto \bigl(\cos(2\cdot t), \sin(2\cdot t)\bigr).\end{aligned}$$
3. The doubling map
$$\begin{aligned}(\mathbb{S}^1, 2\cdot g^\circ) &\longrightarrow (\mathbb{S}^1, g^\circ)\\ (\cos t, \sin t) &\longmapsto \bigl(\cos(2\cdot t), \sin(2\cdot t)\bigr).\end{aligned}$$
4. The map
$$\begin{aligned}(\mathbb{R}^2, g) &\longrightarrow (\mathbb{R}^2, g)\\ (x, y) &\longmapsto (x^2, y^2).\end{aligned}$$

Exercises (for the session on November 30/December 1). The following exercises (which all are solvable with the material read/discussed in week 3) will be discussed.

Please turn over

Exercise 3.1 (Riemannian metrics on $\mathbb{R}^2$)**.** Let g be a Riemannian metric on $\mathbb{R}^2$ and let x^1, x^2 be the standard coordinates on $\mathbb{R}^2$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. There exist $f_{11}, f_{22} \in C^\infty(\mathbb{R}^2)$ with
$$g = f_{11} \cdot (\mathrm{d}\, x^1)^2 + f_{22} \cdot (\mathrm{d}\, x^2)^2.$$

2. There exist $f_{11}, f_{21}, f_{22} \in C^\infty(\mathbb{R}^2)$ with
$$g = f_{11} \cdot (\mathrm{d}\, x^1)^2 - f_{21} \cdot \mathrm{d}\, x^2 \cdot \mathrm{d}\, x^1 + f_{22} \cdot (\mathrm{d}\, x^2)^2.$$

Exercise 3.2 (scaling Riemannian metrics)**.** Let M be a smooth manifold, let $g_1, g_2 \in \operatorname{Riem}(M)$, and let $f_1, f_2 \in C^\infty(M, \mathbb{R}_{\geq 0})$ with the property that $f_1 + f_2 > 0$ (pointwise). Moreover, let

$$\begin{aligned} g := f_1 \cdot g_1 + f_2 \cdot g_2 \colon M &\longrightarrow \mathbf{T}^{2,0} M \\ x &\longmapsto \big(v \otimes w \mapsto f_1(x) \cdot (g_1)_x(v \otimes w) + f_2(x) \cdot (g_2)_x(v \otimes w)\big) \end{aligned}$$

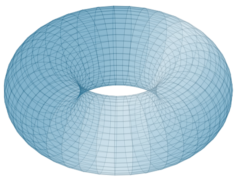
Show that g is a Riemannian metric on M.

Exercise 3.3 (two 2-tori)**.** Let g° denote the round Riemannian metric on $\mathbb{S}^1$.

1. Show that the Riemannian manifold $(\mathbb{S}^1 \times \mathbb{S}^1, g^\circ \oplus g^\circ)$ is locally isometric to $\mathbb{R}^2$ with the Euclidean Riemannian metric.

2. Find a smooth submanifold $T \subset \mathbb{R}^3$ that is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.

3. *Bonus problem.* Is your diffeomorphism isometric with respect to $g^\circ \oplus g^\circ$ and the first fundamental form of T in $\mathbb{R}^3$?



Exercise 3.4 (existence of local orthonormal frames)**.** Let (M, g) be a Riemannian manifold and let $x \in M$. Show that there exists an open neighbourhood $U \subset M$ of x that admits an orthonormal frame.

An *orthonormal frame on* U is a family $(s_1, \dots, s_n)$ of sections $U \longrightarrow \mathrm{T}\, M$ of the tangent bundle of M over U (where $n := \dim M$) with the following property: For each $y \in U$, the family $(s_1(y), \dots, s_n(y))$ in $\mathrm{T}_y\, M$ is an orthonormal basis for $\mathrm{T}_y\, M$ with respect to g_y.

Hints. Gram-Schmidt might help! You should *not* aim at proving that there always exist local orthonormal *coordinate* frames (because this is wrong ...).

Bonus problem (visualisation of Riemannian metrics)**.** Devise a program that visualises Riemannian metrics on $\mathbb{R}^2$:

1. How could one visualise Riemannian metrics on $\mathbb{R}^2$?

   Hints. What about drawing the shape/size of unit circles in the tangent planes at different points of $\mathbb{R}^2$?

2. Implement as much of this as you can.

Submission before November 26, 2020, 10:00, via email to your tutor.

Differential Geometry I: Week 5

Prof. Dr. C. Löh/AG Ammann November 26, 2020

Reading assignment (for the lecture on December 2)**.** We will first see that good actions by symmetries lead to interesting examples of Riemannian manifolds via quotients. Then, we will switch to extracting geometric data and invariants from Riemannian metrics, starting with the length of curves.

- Read Chapter *1.4.5 Group actions*.
- Read Chapter *1.5.1 Lengths of curves*.

Reading assignment (for the lecture on December 3)**.** Using length of curves, we can introduce the Riemannian distance function on Riemannian manifolds.

- Read Chapter *1.5.2 The Riemannian distance function*.
- As a preparation for next week: Recall differential forms on smooth manifolds and their integration.

Next week, we will quickly review orientations and volume and then start developing the analytic foundations for curvature of Riemannian manifolds.

Étude (lengths of curves)**.** Compute the lengths of the following curves:

1. $[0, 2\pi] \longrightarrow \mathbb{S}^2,\ t \longmapsto (\cos t, \sin t, 0)$ in $\mathbb{S}^2$ with the round metric.
2. $[0, 4\pi] \longrightarrow \mathbb{S}^2,\ t \longmapsto (\cos t, \sin t, 0)$ in $\mathbb{S}^2$ with the round metric.
3. $[0, 2\pi] \longrightarrow \mathbb{S}^2(2),\ t \longmapsto 2 \cdot (0, \cos t, \sin t)$ in $\mathbb{S}^2(2)$ with the round metric.
4. $[0, 1] \longrightarrow \mathbb{U}^2(1),\ t \longmapsto (t, t + 1)$ in the Poincaré halfspace model.
5. $[0, 1] \longrightarrow \mathbb{U}^2(2),\ t \longmapsto (t, t + 1)$ in the Poincaré halfspace model.
6. $[0, 2\pi] \longrightarrow \mathbb{U}^2(1),\ t \longmapsto (\cos t, \sin t + 2)$ in the Poincaré halfspace model.

Exercises (for the session on December 7/8)**.** The following exercises (which all are solvable with the material read/discussed in week 4) will be discussed.

Please turn over

Exercise 4.1 (scaled models)**.** Let $n \in \mathbb{N}$ and $R \in \mathbb{R}_{>0}$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. There exists an isometry $(\mathbb{S}^n(R), g_R) \longrightarrow (\mathbb{S}^n(1), R^2 \cdot g_1)$, where g_R denotes the round metric on $\mathbb{S}^n(R)$.

2. There exists an isometry $(\mathbb{H}^n(R), g_R) \longrightarrow (\mathbb{H}^n(1), R^2 \cdot g_1)$, where g_R denotes the standard hyperbolic Riemannian metric on $\mathbb{H}^n(R)$.

Exercise 4.2 (the Möbius transformation action on the hyperbolic plane)**.** We consider the Möbius transformation action

$$\begin{aligned} \mathrm{SL}(2,\mathbb{R}) \times \mathbb{U}^2(1) &\longrightarrow \mathbb{U}^2(1) \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) &\longmapsto \frac{a \cdot z + b}{c \cdot z + d} \end{aligned}$$

on the Poincaré halfspace model $\mathbb{U}^2(1)$ (viewed as a subset of $\mathbb{C}$). Solve two of the following problems:

1. Show that this defines an action on $\mathbb{U}^2(1)$.

2. Show that this action is isometric with respect to $g^{\mathbb{U}}$.

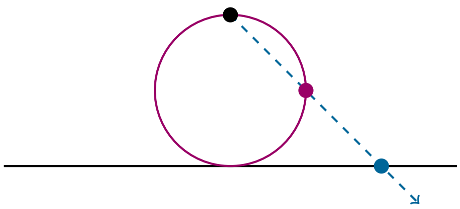
3. Is this action transitive? Justify your answer!

4. What is the stabiliser group at i ?

Exercise 4.3 (spheres are locally conformally flat)**.** Let $n \in \mathbb{N}$ and let g be the Euclidean Riemannian metric on $\mathbb{R}^n$. Show that the round n-sphere $(\mathbb{S}^n, g^\circ)$ is *locally conformally flat*, i.e., that around each point of $\mathbb{S}^n$, there exists a smooth chart $\varphi\colon U \longrightarrow U'$ and a smooth function $f\colon U' \longrightarrow \mathbb{R}_{>0}$ with

$$(\varphi^{-1})^* g^\circ = f^2 \cdot g.$$

Hints. Use homogeneity to reduce the problem to the south pole. Then use the spherical stereographic projection and proceed as in the comparison between the hyperboloid model and the Poincaré disk model. In fact, the formulas will be very similar (just some signs will flip).



Exercise 4.4 (Cayley transform)**.** Let $n \in \mathbb{N}_{>0}$. Show that the Cayley transform

$$\begin{aligned} c\colon\ \mathbb{U}^n(R) &\longrightarrow \mathbb{B}^n(R) \\ \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \ni (x,y) &\longmapsto \frac{1}{\|x\|_2{}^2 + (y+R)^2} \cdot \bigl(2 \cdot R^2 \cdot x, R \cdot (\|x\|_2{}^2 + \|y\|_2{}^2 - R^2)\bigr) \end{aligned}$$

is an isometry $(\mathbb{U}^n(R), g^{\mathbb{U}}) \longrightarrow (\mathbb{B}^n(R), g^{\mathbb{B}})$.

Hints. In order to manage the complexity of this task, you may restrict to the case of $n = 2$ (without using complex analysis, because this would not generalise directly to the higher-dimensional case) or resort to the help of computer algebra systems (but your submission has to be comprehensible and verifiable for a human reader).

Bonus problem (hyperbolic art)**.** Sketch how Escher's *Cirkellimiet IV* would look like in the Poincaré halfplane model!

Submission before December 3, 2020, 10:00, via email to your tutor.

Differential Geometry I: Week 6

Prof. Dr. C. Löh/AG Ammann December 3, 2020

Reading assignment (for the lecture on December 9). In addition to lengths, we can also consider volumes. Moreover, we will take the first step towards the introduction of analytic notions of curvature.

- Read Chapter 1.5.3 *Volume and orientation.*

  We will use volumes only sparsely in this course. This is why the treatment is mostly just a short recap of material that most of you might have seen before. If you don't know anything about differential forms, this will probably not be much of a problem for the rest of the course.

- Read Chapter 2.1 *The idea of curvature.*

Reading assignment (for the lecture on December 10). We now formalise the notion of connections. Connections will be extremely important for the rest of this course; thus, it is best to become good friends with them as soon as possible!

- Read Chapter 2.2.1 *Connections.*
- Read Chapter 2.2.2 *Local descriptions of connections.*

Next week, we will apply connections to vector fields along curves, introduce geodesics and parallel transport. Finally, we will add compatibility with Riemannian metrics to the picture.

Étude (the Euclidean connection). Let $\overline{\nabla}$ denote the Euclidean connection on $\mathbb{R}^2$, let $X := ((x,y) \mapsto (1,0))$, let $Y := ((x,y) \mapsto (0,1))$, and let $Z := ((x,y) \mapsto (x^2, x \cdot y^3))$.

1. Show the property (FL1) for $\overline{\nabla}$.
2. Show the property (L2) for $\overline{\nabla}$.
3. Show the property (F2) for $\overline{\nabla}$.
4. Compute $\overline{\nabla}_X Y$ and $\overline{\nabla}_X X$.
5. Compute $\overline{\nabla}_X Z$ and $\overline{\nabla}_Z X$.
6. Compute $\overline{\nabla}_Z Z$.

Exercises (for the session on December 14/15). The following exercises (which all are solvable with the material read/discussed in week 5) will be discussed.

Please turn over

Exercise 5.1 (Heisenberg group)**.** We consider the subgroup (*Heisenberg group*)

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\} \subset \mathrm{SL}(3, \mathbb{Z})$$

of $\mathrm{SL}(3, \mathbb{Z})$ and its action by matrix multiplication on $\mathbb{R}^3$. Which of the following statements are true? Justify your answer!

1. The action is isometric with respect to the Euclidean Riemannian metric.
2. The action is free and proper.

Exercise 5.2 (Klein bottle)**.** We consider the following action

$$\begin{aligned} (\mathbb{Z} \rtimes \mathbb{Z}) \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \big((n_1, n_2), (x_1, x_2)\big) &\longmapsto \big((-1)^{n_2} \cdot x_1 + n_1, x_2 + n_2\big) \end{aligned}$$

on $\mathbb{R}^2$, where $\mathbb{Z} \rtimes \mathbb{Z}$ is defined via the action of $1 \in \mathbb{Z}$ by multiplication by -1 on $\mathbb{Z}$.

1. Show that this action is isometric (with respect to the Euclidean Riemannian metric), proper, and free. The Riemannian quotient of this action is the *Klein bottle*.
2. Show that there exists an action of $\mathbb{Z}/2$ on a suitably scaled 2-torus T such that the quotient $(\mathbb{Z}/2) \setminus T$ with the quotient Riemannian metric is isometric to the Klein bottle.

Exercise 5.3 (the punctured plane)**.** We consider $M := \mathbb{R}^2 \setminus \{0\}$ with the smooth structure induced by $\mathbb{R}^2$ and the Euclidean Riemannian metric g. Let $x := (1, 0)$, $y := (-1, 0) \in M$.

1. Compute $d_g(x, y)$! Justify your answer!
2. Is there a piecewise regular curve $\gamma \colon [0, 1] \longrightarrow M$ with $\gamma(0) = x$ and $\gamma(1) = y$ as well as $L_g(\gamma) = d_g(x, y)$? Justify your answer in detail and illustrate your arguments with suitable pictures!

Exercise 5.4 (quotients of symmetric spaces)**.** Let (M, g) be a symmetric space and let $\Gamma \curvearrowright (M, g)$ be an isometric, free, and proper action of a discrete group Γ. Show that the quotient $\Gamma \setminus M$ (with the induced smooth structure and Riemannian metric) is a locally symmetric space.

Hints. Use the Riemannian distance function to find invariant neighbourhoods in the quotient.

Bonus problem (locally symmetric spaces from lattices)**.**

1. Look up the terms *Lie group* and *lattice in a Lie group* in the literature.
2. How do lattices in Lie groups lead to locally symmetric spaces?

   Hints. Giving a precise formulation of such a statement (and a reference) and briefly explaining the terms/construction is enough.

Submission before December 10, 2020, 10:00, via email to your tutor.

Differential Geometry I: Week 7

Prof. Dr. C. Löh/AG Ammann December 10, 2020

Reading assignment (for the lecture on December 16). Using linear connections, we can differentiate vector fields along curves. In particular, this allows to introduce an analytic notion of geodesics.

- Read Chapter 2.2.3 *Covariant derivatives along curves.*
- Recall basics on ordinary differential equations (in particular: existence and uniqueness theorems for solutions of ordinary differential equations).
- Read Chapter 2.2.4 *Geodesics* until Theorem 2.2.22.

Reading assignment (for the lecture on December 17). Covariant derivatives along curves give also rise to parallel transport of tangent vectors along curves. This parallel transport provides a nice, geometric, reinterpretation of covariant derivatives and connections.

- Read the rest of Chapter 2.2.4 *Geodesics.*
- Read Chapter 2.2.5 *Parallel transport.*
- Read the introduction of Chapter 2.3 *The Levi-Civita connection.*

Étude (covariant derivatives along curves). Let

$$\begin{aligned}
\gamma\colon \mathbb{R} &\longrightarrow \mathbb{R}^2, & t &\longmapsto (\cos t, \sin t)\\
X\colon \mathbb{R} &\longrightarrow \mathbb{R}^2, & t &\longmapsto (-t\cdot \sin t, t\cdot \cos t)\\
Y\colon \mathbb{R} &\longrightarrow \mathbb{R}^2, & t &\longmapsto (42, 0)\\
Z\colon \mathbb{R} &\longrightarrow \mathbb{R}^2, & t &\longmapsto (t, t^{2020}).
\end{aligned}$$

Which of the terms

$$D_\gamma X, \qquad D_\gamma Y, \qquad D_\gamma Z$$

make sense

- with respect to the Euclidean linear connection on $\mathbb{R}^2$?
- with respect to the linear connection on $\mathbb{S}^1$ induced by the Euclidean linear connection on $\mathbb{R}^2$?

In case they do make sense, compute them explicitly.

Exercises (for the session on December 21/22). The following exercises (which all are solvable with the material read/discussed in week 6) will be discussed.

Please turn over

Exercise 6.1 (volumes). Let M be a compact non-empty smooth manifold, let g_1, g_2 be Riemannian metrics on M, and let $\lambda \in \mathbb{R}_{>0}$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

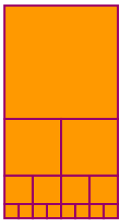
1. If $g_2 = \lambda^2 \cdot g_1$, then $\operatorname{vol}(M, g_2) = \lambda^n \cdot \operatorname{vol}(M, g_1)$.
2. If $\operatorname{vol}(M, g_2) = \lambda^n \cdot \operatorname{vol}(M, g_1)$, then $g_2 = \lambda^2 \cdot g_1$.

Exercise 6.2 (volume growth of hyperbolic spaces). Let $n \in \mathbb{N}_{\geq 2}$. We equip the halfspace $\mathbb{U}^n := \mathbb{U}^n(1)$ with the metric $g^{\mathbb{U}}$ (of radius 1). Show that there exist constants $a \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}_{>0}$ such that for each $x \in \mathbb{U}^n$, we have

$$\forall_{r \in \mathbb{R}_{>0}} \quad \varrho_x^{(\mathbb{U}^n, g^{\mathbb{U}})}(r) \geq c \cdot r^{n-1} \cdot a^r.$$

Hints. Look at the point $(0, \dots, 0, 1)$ and sets of the form $\{(x_1, \dots, x_{n-1}, y) \in \mathbb{U}^n \mid y \in [e^{-r/(2n)}, 1],\ x_1, \dots, x_{n-1} \in [0, r/(2n)]\}$; then estimate the Riemannian distance function and the Riemannian volume in the correct direction.

Bonus problem. How is this related to the picture below?



Exercise 6.3 (construction of connections). Solve two of the following problems:

1. Show that the pullback of a linear connection is a linear connection (Proposition 2.2.4).
2. Show that the induced connection on submanifolds is well-defined (Example 2.2.8).
3. Show that the induced connection on submanifolds is a connection (Example 2.2.8).
4. Show Proposition 2.2.11 on restrictions of connections.

Exercise 6.4 (connection coefficients). Compute the connection coefficients of the Euclidean linear connection on $\mathbb{R}^2 \setminus \{0\}$ with respect to polar coordinates.
Hints. This requires some careful bookkeeping … Don't panic!

Bonus problem (the affine space of connections). Let M be a smooth manifold and let ∇ be a linear connection on M. Show that the set of all linear connections on M can alternatively be described as

$$\bigl\{ \text{“}\nabla + D\text{”} \bigm| D \in \Gamma(\mathbf{T}^{(1,2)} M) \bigr\}.$$

Hints. You should first give a sensible interpretation for the term “$\nabla + D$”.

Submission before December 17, 2020, 10:00, via email to your tutor.

Differential Geometry I: Week 8

Prof. Dr. C. Löh/AG Ammann December 17, 2020

Reading assignment (for the lecture on December 23). We now let connections and Riemannian metrics interact. In particular, we will formulate the *fundamental theorem of Riemannian geometry* (sounds important!).

- Recall the concept of tensor bundles of the tangent bundle (Chapter 1.2.4)
- Read Chapter 2.3.1 *Connections on tensor bundles.*
- Recall the notion of *Lie brackets* of vector fields.
- Read Chapter 2.3.2 *The Levi-Civita connection* until Theorem 2.3.17 (without proof).

  The technicalities of the proofs of the alternative characterisations of compatibility of a connection with a metric and of symmetry are a good opportunity to practice the various notions and constructions, but they are not that important in the long run. Thus, for now, it is sufficient to have a quick glance at these computations.

I wish you a restful Christmas break and a good start into the year 2021 !

Reading assignment (for the lecture on January 7). Welcome back!

- Recall the terms Riemannian manifolds, connections, geodesics, and parallel transport.
- Recall the defining properties of the Levi-Civita connection.
- Read the rest of Chapter 2.3.2 *The Levi-Civita connection.*
- Read (the beginning of) Chapter 2.4.1 *The Riemannian curvature tensor.*

Next week, we will study first properties of the Riemannian curvature tensor and its siblings.

Étude (decrypting differential geometry). Find the correct bijection between terms (with the "obvious" implicit meaning of all symbols) and descriptions!

$\nabla_{E_i} E_j = \sum_k {\Gamma_{ij}}^k \cdot E_k$	Riemannian metric of the hyperbolic halfspace model
$\sum_\ell ({\Gamma_{ki}}^\ell \cdot g_{\ell j} + {\Gamma_{kj}}^\ell \cdot g_{i\ell}) = E_k(g_{ij})$	parallel transport equation
$\frac{4}{1-\lVert x\rVert_2{}^2} \cdot \sum_i^n (\mathrm{d}\, x^i)^2$	compatibility of a connection with a Riemannian metric
$x^{k\prime\prime} + \sum_i \sum_j x^{i\prime} \cdot x^{j\prime} \cdot {\Gamma_{ij}}^k \circ (x^1, \dots, x^n) = 0$	zero
$X^{k\prime} + \sum_i \sum_j X^i \cdot \gamma^{j\prime} \cdot \Gamma_{ij} \circ \gamma = 0$	defining equation for connection coefficients
$\frac{2020}{2021} \cdot (g_{ij} - g_{ji})$	geodesic equation

Exercises (for the session on January 11/12). The following exercises (which all are solvable with the material read/discussed in week 7) will be discussed.

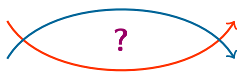
Please turn over

Exercise 7.1 (reparametrising geodesics)**.** Let M be a smooth manifold with a linear connection ∇ and let $\gamma\colon \mathbb{R} \longrightarrow M$ be a smooth curve. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If γ is a geodesic with respect to ∇, then also $t \longmapsto \gamma(-2020 \cdot t)$ is a geodesic with respect to ∇.
2. If γ is a geodesic with respect to ∇, then also $t \longmapsto \gamma(t^{2020})$ is a geodesic with respect to ∇.

Exercise 7.2 (maximal geodesics; Corollary 2.2.24)**.** Let M be a smooth manifold with a linear connection ∇ and let $x \in M$.

1. Show that for each $v \in \mathrm{T}_x M$ there exists a unique maximal geodesic $\gamma\colon I \longrightarrow M$ with $0 \in I^\circ$ and
$$\gamma(0) = x \quad \text{and} \quad \dot\gamma(0) = v.$$
2. Let $y \in M$ with $x \neq y$. Can it happen that there are two different maximal geodesics $\gamma, \eta\colon I \longrightarrow M$ with $\gamma(0) = x = \eta(0)$ and $\gamma(1) = y = \eta(1)$? Justify your answer with a suitable proof or counterexample!



Exercise 7.3 (connections via parallel transport; Corollary 2.2.35)**.** Let M be a smooth manifold with a linear connection ∇, let $X, Y \in \Gamma(\mathrm{T}\,M)$, and let $x \in M$. Moreover, let $\gamma\colon I \longrightarrow M$ be a smooth curve in M with $0 \in I$ and
$$\gamma(0) = x \quad \text{and} \quad \dot\gamma(0) = X(x),$$
Show that
$$(\nabla_X Y)(x) = \lim_{h \to 0} \frac{P^\gamma_{h,0}(Y(\gamma(h)) - Y(x)}{h}.$$

Exercise 7.4 (geodesics and parallel transport)**.** On $\mathbb{R}^3$, we consider the linear connection ∇ that is given by
$$\begin{array}{lll} \nabla_X X = 0 & \nabla_Y X = -Z & \nabla_Z X = Y \\ \nabla_X Y = Z & \nabla_Y Y = 0 & \nabla_Z Y = -X \\ \nabla_X Z = -Y & \nabla_Y X = X & \nabla_Z Z = 0 \end{array}$$
in terms of the standard coordinate frame (X, Y, Z) of $\mathrm{T}\,\mathbb{R}^3$.

1. Show that the maximal geodesics with respect to ∇ are exactly the constant speed affine lines.
2. We consider $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}^3, t \longmapsto (t, 0, 0)$. Compute the parallel transport $P^\gamma_{0,1}\colon \mathrm{T}_{\gamma(0)}\,\mathbb{R}^3 \longrightarrow \mathrm{T}_{\gamma(1)}\,\mathbb{R}^3$ and illustrate the result suitably!

Bonus problem (SageMath)**.**

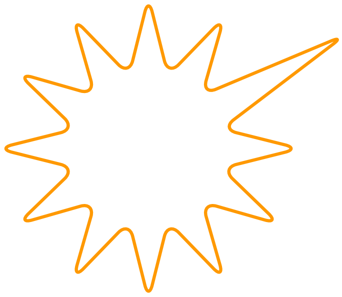
1. Install SageMath (https://www.sagemath.org).
2. Use SageMath to compute the connection coefficients of the Euclidean linear connection on $\mathbb{R}^2$ with respect to polar coordinates and document the individual steps.

Submission before January 7, 2021, 10:00, via email to your tutor.

Differential Geometry I: Week $8\frac{1}{2}$

Prof. Dr. C. Löh/AG Ammann December 17, 2020

Bonus problem (one-dimensional Riemannian geometry is boring)**.** Show that every Riemannian manifold of dimension 1 is locally isometric to $\mathbb{R}$ (equipped with the Euclidean Riemannian metric).



Bonus problem (rescaling maximal geodesics)**.** Let M be a smooth manifold with a linear connection ∇, let $x \in M$, and let $v \in \mathrm{T}_x M$. Show that for all $c \in \mathbb{R}$ and all $t \in \mathbb{R}$, we have

$$\mathrm{geod}_{x,c\cdot v}(t) = \mathrm{geod}_{x,v}(c \cdot t),$$

whenever either side is defined. Here, we consider the maximal geodesics with respect to ∇.

Bonus problem (covariant derivatives along curves on submanifolds)**.** Let $N \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a smooth submanifold. We equip M with the linear connection $\nabla^\top$ induced (via the orthogonal projection $p \colon \mathrm{T}\,\mathbb{R}^N \longrightarrow \mathrm{T}\,M$) from the Euclidean linear connection $\overline{\nabla}$ on $\mathbb{R}^N$. Moreover, we denote the corresponding covariant derivatives along curves by $D^\top$ and $\overline{D}$, respectively.

Let $\gamma \colon I \longrightarrow M$ be a smooth curve. Show that

$$\forall_{X \in \Gamma(\mathrm{T}\,M|_\gamma)} \quad \forall_{t \in I} \quad (D^\top_\gamma X)(t) = p\bigl((\overline{D}_\gamma X)(t)\bigr).$$

Hints. If you like local coordinates, you can just perform this computation in local coordinates. If you want to avoid local coordinates, you can work with the defining properties of covariant derivatives along curves.

Bonus problem (curves)**.** Explain as many terms as possible related to curves and vector fields using rollercoasters. Draw pictures!
Hints. Don't waste too much time on watching videos on crazy rollercoasters!

Bonus problem (lecture notes)**.** Find typos in the lecture notes!

Optional submission before January 7, 2021, 10:00, via email to your tutor.

Differential Geometry I: Week 9

Prof. Dr. C. Löh/AG Ammann January 7, 2021

Reading assignment (for the lecture on January 13). We continue the study of the Riemannian curvature tensor. In particular, we characterise flat manifolds.

- Read the rest of Chapter 2.4.1 *The Riemannian curvature tensor.*
- Read Chapter 2.4.2 *Flat manifolds.*

Reading assignment (for the lecture on January 14). From the Riemannian curvature tensor, we extract various degrees of information. Moreover, we start with the computation of the curvatures of model spaces.

- Read Chapter 2.4.3 *Sectional curvature.*
- Read Chapter 2.4.4 *Ricci curvature.*
- Read Chapter 2.4.5 *Scalar curvature.*
- Read Chapter 2.5.1 *Locally conformally flat manifolds.*

Next week, we will complete the curvature computations for model spaces and we will start working with Riemannian geodesics.

Étude (tensors). Let M be a smooth manifold and let ∇ be a linear connection on M. Which of the following terms define tensor fields on M ? Here, $X, Y, \dots$ denote vector fields on M.

1. $X \longmapsto \nabla_X X$
2. $(X, Y) \longmapsto \nabla_X Y$
3. $(X, Y) \longmapsto \nabla_X Y - \nabla_Y X$
4. $(X, Y) \longmapsto [X, Y]$
5. $(X, Y, Z) \longmapsto \nabla_{[X,Y]} Z$
6. $(X, Y, Z) \longmapsto \nabla_{[Y,Z]} X + \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X$

Exercises (for the session on January 18/19). The following exercises (which all are solvable with the material read/discussed in week 8) will be discussed.

Please turn over

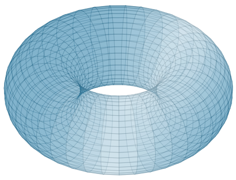
Exercise 8.1 (scaled manifolds)**.** Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let $\lambda \in \mathbb{R}_{>0}$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. The Levi-Civita connection of $(M, \lambda \cdot g)$ is $\lambda \cdot \nabla$.
2. The Riemannian curvature $(3, 1)$-tensor of $(M, \lambda \cdot g)$ coincides with the Riemannian curvature $(3, 1)$-tensor of (M, g).

Exercise 8.2 (the return of the torus)**.** We consider the 2-torus $T \subset \mathbb{R}^3$, given by

$$\bigl\{((2 + \cos x) \cdot \cos y, (2 + \cos x) \cdot \sin y, \sin x) \bigm| x, y \in \mathbb{R}\bigr\} \subset \mathbb{R}^3$$

and equipped with the Riemannian metric induced by the Euclidean Riemannian metric on $\mathbb{R}^3$. Show that the Riemannian curvature $(3, 1)$-tensor of T is *not* everywhere zero.



Exercise 8.3 (tensor characterisation lemma; Proposition 2.3.2)**.** Prove the tensor characterisation lemma.

Exercise 8.4 (compatible connections; Proposition 2.3.7)**.** Let (M, g) be a Riemannian manifold, let ∇ be a linear connection on M and suppose that the parallel transport maps with respect to ∇ are isometries. Show that for all smooth curves γ on M and all $X, Y \in \Gamma(\mathrm{T}\, M|\gamma)$, we have

$$\langle X, Y\rangle_g' = \langle D_\gamma X, Y\rangle_g + \langle X, D_\gamma Y\rangle_g.$$

Hints. Show that parallel orthonormal frames exist and use them.

Bonus problem (commuting vector fields)**.**

1. What is the flow of a vector field? What can you say about existence of flows of vector fields?
2. Prove the following realisation theorem on commuting vector fields: Let M be a smooth manifold of dimension n and let $E_1, \dots, E_n \in \Gamma(\mathrm{T}\, M)$ be smooth vector fields on M with the following properties:
   - The family $(E_1, \dots, E_n)$ is a frame.
   - For all $j, k \in \{1, \dots, n\}$, we have $[E_j, E_k] = 0$.

   Show that for each $x \in M$, there exists an open neighbourhood U around x such that $(E_j|_U)_{j\in\{1,\dots,n\}}$ is a *coordinate* frame on U.

   Hints. Use iterated flows to find a suitable chart.

Submission before January 14, 2021, 10:00, via email to your tutor.

Differential Geometry I: Week 10

Prof. Dr. C. Löh/AG Ammann January 14, 2021

Reading assignment (for the lecture on January 20)**.** We will now use symmetry to compute the sectional curvature of the model spaces. Moreover, we start with the exploration of Riemannian geodesics.

- Read Chapter 2.5.2 *Symmetries and constant curvature.*
- Recall the construction of the *model spaces* and their basic properties.
- Read Chapter 2.5.3 *Sectional curvature of the model spaces.*
- Recall the notion of *geodesic* (including the geodesic equation) and of *maximal geodesics.*
- Read Chapter 3.1.1 *The exponential map.*

Reading assignment (for the lecture on January 21)**.** Our next goal is to understand the relation between Riemannian and metric geodesics, using a variational approach.

- Read Chapter 3.1.2 *Normal coordinates.*
- Recall the notion of *piecewise regular curves* and *length of curves.*
- Read Chapter 3.2.1 *Variation of curves.*
- Read Chapter 3.2.2 *Variation fields and the first variation formula.*

Étude (using curvature)**.** Which of the following manifolds are isometric? Locally isometric?

1. $\mathbb{S}^1(1)$
2. $\mathbb{S}^1(2021)$
3. $\mathbb{S}^{2021}(1)$
4. $\mathbb{S}^{2021}(2020)$
5. $\mathbb{R}^1$
6. $\mathbb{R}^{2021}$

Exercises (for the session on January 25/26)**.** The following exercises (which all are solvable with the material read/discussed in week 9) will be discussed.

Please turn over

Exercise 9.1 (flat manifolds?). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. The cylinder $\mathbb{S}^1 \times \mathbb{R}$ (with the product metric of the round metric and the Euclidean Riemannian metric) is flat.

2. The Klein bottle admits a flat Riemannian metric.

3. *Bonus problem (if you know Algebraic Topology).* The torus $\mathbb{S}^1 \times \mathbb{S}^1$ and the Klein bottle admit Riemannian metrics that make them isometric?!

Exercise 9.2 (one-argument Ricci curvature; Remark 2.4.27). Let (M, g) be a Riemannian manifold.

1. Show that Ric is symmetric in its two arguments.

2. Show that Ric can be recovered from only knowing the map

$$\begin{aligned} \mathrm{T}\,M &\longrightarrow \mathbb{R} \\ \mathrm{T}_x\,M \ni v &\longmapsto \mathrm{Ric}_x(v, v). \end{aligned}$$

Exercise 9.3 (sectional curvature determines Riemannian curvature; Proposition 2.4.23). Let M be a smooth manifold and let R_1, R_2 be $(4, 0)$-tensor fields on M that satisfy the symmetries in Proposition 2.4.9. Moreover, for all $x \in M$ and all linearly independent $v, w \in \mathrm{T}_x\,M$ we assume that

$$R_1(v, w, w, v) = R_2(v, w, w, v).$$

Show that then $R_1 = R_2$ follows.
Hints. Look at $R_1 - R_2$.

Exercise 9.4 (Riemannian curvature and conformal changes; Theorem 2.5.5). Prove two of the first three claims of Theorem 2.5.5.

Bonus problem (positive curvature).

1. Give a reasonable definition of "positive Ricci curvature".

2. Give an example of a Riemannian manifold that has positive Ricci curvature but that does *not* have positive sectional curvature.

   Hints. You should not search for such examples in dimensions 1, 2, 3.

Submission before January 21, 2021, 10:00, via email to your tutor.

Differential Geometry I: Week 11

Prof. Dr. C. Löh/AG Ammann January 21, 2021

In the next weeks, we will cover a lot of material. The underlying ideas are always geometrically simple and the results that we will obtain are definitely worth it! When reading, it is recommended to first focus on general lines of arguments and only then to delve into the technical details.

Reading assignment (for the lecture on January 27)**.** We determine the relation between minimising curves and geodesics, with the help of the first variation formula.

- Recall the *first variation formula.*
- Recall the definition of the *Riemannian distance function.*
- Read Chapter 3.2.3 *Minimising curves are geodesics.*
- Read Chapter 3.2.4 *Geodesics are locally minimising.*

Reading assignment (for the lecture on January 28)**.** The previous results suggest a close relation between metric and Riemannian geometric properties. We clarify this for the notion of "completeness". Moreover, we apply the techniques for geodesics in the case of model spaces and thus to classical geometric problems.

- Read Chapter 3.2.5 *Riemannian isometries.*
- Read Chapter 3.3.1 *Two notions of completeness.*
- Read Chapter 3.3.2 *The Hopf-Rinow theorem.*
- Read Chapter 3.4.1 *Geodesics of the model spaces.*
- (Optional) Read Chapter 3.4.2 *Isometries of the model spaces.*

Next week, as a preparation for the endgame of this course, we will take our handling of geodesics to the next level by studying variations of geodesics through geodesics. This will open the door to local-global results.

Étude (domains of the exponential map)**.** For each of the following manifolds (with the Riemannian Euclidean metric) and each of the given points, determine the domain of the exponential map at this point. Draw pictures!

Manifold	Points
$\mathbb{R}^2 \setminus \{0\}$	$(-1,0),\ (-2,0)$
$\mathbb{R}^2 \setminus \{0,(2,0)\}$	$(-1,0),\ (1,0)$
$\mathbb{R}^2 \setminus (\mathbb{R}_{>0} \times \{0\})$	$(-1,0),\ (1,1)$

Exercises (for the session on February 1/2)**.** The following exercises (which all are solvable with the material read/discussed in week 10) will be discussed.

The exercise series of next week (i.e., Exercises 11.1–11.4) will be the last regular exercise sheet of this course. All later exercises will count as bonus exercises.

Please turn over

Exercise 10.1 (domain of the exponential map). Let (M, g) be a Riemannian manifold and let $x, y \in M$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If $y \in \exp_x(\mathrm{Exp}_x)$, then $x \in \exp_y(\mathrm{Exp}_y)$.
2. If $\exp_x(\mathrm{Exp}_x) = \exp_y(\mathrm{Exp}_y)$, then $x = y$.

Exercise 10.2 (constant sectional curvature). Let (M, g) be a Riemannian manifold that has constant sectional curvature $c \in \mathbb{R}$. Show that then

$$\mathrm{R}(X, Y, Z) = c \cdot \bigl(\langle Y, Z\rangle_g \cdot X - \langle X, Z\rangle_g \cdot Y\bigr)$$

holds for all $X, Y, Z \in \Gamma(\mathrm{T}\, M)$.

Exercise 10.3 (negative sectional curvature). Let (M, g) be a Riemannian manifold with $\sec < 0$.

1. If M is compact, show that there exists a $\kappa \in \mathbb{R}_{>0}$ with $\sec \leq -\kappa$.
2. If M is non-compact, show that such a bound does not necessarily exist.
3. *Bonus problem.* What happens in the non-compact, connected case? Justify your answer!

Exercise 10.4 (fixed sets are geodesic; Proposition 3.4.1). Let (M, g) be a Riemannian manifold and let $N \subset M$ be a connected one-dimensional smooth submanifold for which there exists a $\varphi \in \mathrm{Isom}(M, g)$ with

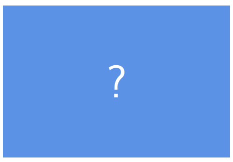
$$N = \bigl\{x \in M \bigm| \varphi(x) = x\bigr\}.$$

Moreover, let $x \in N$ and let $v \in \mathrm{T}_x\, N \setminus \{0\}$. Show that

$$\mathrm{im}(\mathrm{geod}_{x,v}) = N.$$

Hints. First show that $\varphi \circ \mathrm{geod}_{x,v} = \mathrm{geod}_{x,v}$ and then apply a connectedness/extension argument.

Bonus problem (UN flag). What does the UN flag have to do with differential geometry? Explain!



Submission before January 28, 2021, 10:00, via email to your tutor.

Differential Geometry I: Week 12

Prof. Dr. C. Löh/AG Ammann January 28, 2021

Reading assignment (for the lecture on February 3). We will now study the second derivative of the length functional of curves and Jacobi fields. Again, the calculations might look discouraging, but the underlying strategies are straightforward.

- Recall the *first variation formula.*
- Read Chapter 3.5.1 *The second variation formula and the index form.*
- Read Chapter 3.5.2 *Jacobi fields.*

Reading assignment (for the lecture on February 4).

- Read Chapter 3.5.3 *Conjugate points.*
- (Optional) Recall the Gauß-Bonnet theorem.
- (Optional) Recall the Euler characteristic and its properties.
- Read Chapter 4.1 *Local-global results.*
- Recall the *Gauß lemma.*
- Read Chapter 4.2 *Constant sectional curvature, locally.*

In the final week, we will put the theory developed in this course to good use: We will establish several local-global results.

Étude (variations). Let (M, g) be a Riemannian manifold, let $x \in M$ and let $v \in \mathrm{T}_x M$. We consider the following expressions (on sufficiently small rectangles in normal neighbourhoods of x). Which of them are variations through geodesics? What can you say about their variation fields? Draw pictures!

1. $(s,t) \longmapsto \exp_x(t \cdot v)$
2. $(s,t) \longmapsto \exp_x(s \cdot v)$
3. $(s,t) \longmapsto \exp_x(s \cdot t \cdot v)$
4. $(s,t) \longmapsto \exp_x((s+t) \cdot v)$
5. $(s,t) \longmapsto \exp_x(s^2 \cdot t \cdot v)$
6. $(s,t) \longmapsto \exp_x(s \cdot t^2 \cdot v)$

Exercises (for the session on February 1/2). The following exercises (which all are solvable with the material read/discussed in week 11) will be discussed.

This is the last regular exercise sheet of this course. All later exercises will count as bonus exercises.

Please turn over

Exercise 11.1 (inheriting completeness?)**.** Let $\varphi\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be a local isometry of Riemannian manifolds. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If (M_1, g_1) is complete, then (M_2, g_2) is complete.
2. If (M_2, g_2) is complete, then (M_1, g_1) is complete.

Exercise 11.2 (the unit radial vector field in normal coordinates)**.** Let (M, g) be a Riemannian manifold, let $x \in M$ and let $U \subset M$ be a normal neighbourhood centred at x.

1. Express the radial distance function ϱ and the unit radial vector field $\partial/\partial\varrho$ centred at x in normal coordinates (and supply proofs).
2. Let $y \in U \setminus \{x\}$. Show that

$$\mathrm{d}_y\, \varrho\Big(\frac{\partial}{\partial \varrho}(y)\Big) = 1.$$

Exercise 11.3 (orthogonal decomposition of vector fields along curves)**.** Let (M, g) be a Riemannian manifold, let γ be a unit speed geodesic on M, and let V be a smooth vector field along γ. Moreover, let

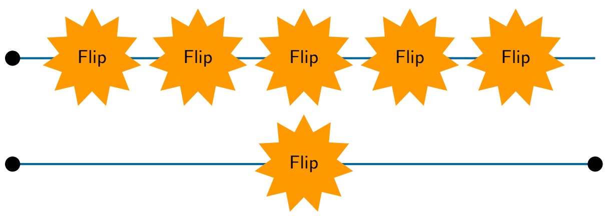
$$\overline{V} := \langle V, \dot\gamma\rangle_g \cdot \dot\gamma \quad \text{and} \quad V^\perp := V - \overline{V}.$$

Show the following compatibility statements (and understand their geometric meaning):

$$\begin{aligned} D_\gamma \overline{V} &= \overline{D_\gamma V} \\ D_\gamma (V^\perp) &= (D_\gamma V)^\perp \\ \|D_\gamma V\|_g^2 &= \langle D_\gamma V, \dot\gamma\rangle_g^2 + \big\|(D_\gamma V)^\perp\big\|_g^2 \\ \mathrm{Rm}(V, \dot\gamma, \dot\gamma, V) &= \mathrm{Rm}(V^\perp, \dot\gamma, \dot\gamma, V^\perp). \end{aligned}$$

Exercise 11.4 (symmetric spaces)**.**

1. Show that every symmetric space is complete.
2. Show that every symmetric space is homogeneous.



Bonus problem (coverings from geometry)**.** Let $\varphi\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be a local isometry of Riemannian manifolds and let M_2 be connected.

1. Show that φ is a covering map if (M_1, g_1) is complete.
2. Does this also hold if (M_1, g_1) is not complete? Justify your answer!

Submission before February 4, 2021, 10:00, via email to your tutor.

Differential Geometry I: Week 13

Prof. Dr. C. Löh/AG Ammann February 4, 2021

Some of the local-global results require a background in algebraic topology. If you don't have this background, then just view these results as an outlook on further topics (without the obligation to understand all the details, but maybe as a motivation to attend a course on algebraic topology in the future).

Reading assignment (for the lecture on February 10)**.** Using Jacobi fields, we will establish several comparison results; this will also lead to a classification of complete Riemannian manifolds with constant sectional curvature.

- Read Chapter 4.3 *Analytic and geometric comparison theorems.*
- Recall the notion of *covering maps.*
- Read Chapter 4.4.1 *The Cartan-Hadamard theorem.*
- Read Chapter 4.4.2 *Constant non-positive sectional curvature.*

Reading assignment (for the lecture on February 11)**.**

- Read Chapter 4.5.1 *The Bonnet-Myers theorem.*
- Read Chapter 4.5.2 *Constant positive sectional curvature.*
- Read Chapter 4.6 *The Švarc-Milnor lemma.*

This is it!

Étude (theorems)**.** Recall the statements and proof ideas of these theorems:

1. Bonnet-Myers
2. Cartan-Hadamard
3. Gauß lemma
4. Hopf-Rinow

Please turn over

Exercise 12.1 (constant sectional curvature)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. The smooth manifold $\mathbb{R}^{2021}$ admits a Riemannian metric of constant positive sectional curvature.
2. The smooth manifold $\mathbb{S}^1 \times \mathbb{R}^{2021}$ admits a Riemannian metric of constant negative sectional curvature.

Exercise 12.2 (conjugate points vs. minimising)**.** Show that the cylinder $\mathbb{R} \times \mathbb{S}^1$ with the product metric of the Euclidean Riemannian metric and the round metric (of radius 1) has the following property: There exist geodesics without conjugate points that are *not* minimising.

Exercise 12.3 (Jacobi fields along radial geodesics; Proposition 4.2.3)**.** Let (M, g) be a Riemannian manifold of dimension n, let $x \in M$, let $U \subset M$ be a normal neighbourhood of x, and let $(E_j)_{j\in\{1,\dots,n\}}$ be the coordinate frame associated with some normal coordinates on U, centred at x. Let $\gamma\colon [0, b] \longrightarrow U$ be a radial geodesic starting at x and let $v = \sum_{j=1}^n v^j \cdot E_j(x) \in \mathrm{T}_x M$. Show that then the Jacobi field V along γ with

$$V(0) = 0 \quad \text{and} \quad D_\gamma V(0) = v$$

is given by

$$\begin{aligned} V\colon [0, b] &\longrightarrow \mathrm{T}\, M \\ t &\longmapsto t \cdot \sum_{j=1}^n v^j \cdot E_j\bigl(\gamma(t)\bigr). \end{aligned}$$

Exercise 12.4 (volume growth in negative curvature; Corollary 4.4.5)**.** Let (M, g) be a Riemannian manifold that satisfies $\sec \le c$ for some $c \in \mathbb{R}_{<0}$. Moreover, let $x \in M$.

1. Let $n \in \mathbb{N}$ and let $A, B \in M_{n\times n}(\mathbb{R})$ be symmetric positive definite matrices with
$$\forall_{x\in\mathbb{R}^n} \quad x^\mathsf{T} \cdot A \cdot x \ge x^\mathsf{T} \cdot B \cdot x.$$
Show that $\det A \ge \det B$.

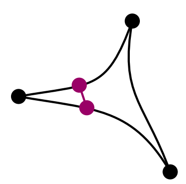
   Hints. Principal axis theorem (Hauptachsentransformation) or Courant minimax principle.

2. Show that (M, g) has (at least) exponential growth at x.

   Hints. What about normal neighbourhoods and radial distances? Apply the metric comparison theorem, the first part, and Exercise 6.2.

Bonus problem (comparison geometry)**.**

1. Look up *Toponogov's theorem* in the literature.
2. Look up the notion of $\mathrm{CAT}(\kappa)$*-spaces* in the literature.

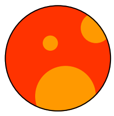


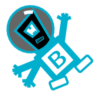
All these exercises count as bonus exercises. Optional submission!

Differential Geometry I: Week 14

Prof. Dr. C. Löh/AG Ammann February 11, 2021

CAT(2021). Commander Blorx is invited as a special guest to the **Congress of Astro-Topologists 2021** . Blorx feels quite honoured, but is also a bit afraid that there might be too many too uninspiring talks and not enough chocolate cookies. This year's hot topic of the congress is planetary design products. Help Blorx to get through this event!





Problem 13.1 (the opening anthem)**.** Same procedure as every year: The congress begins with the opening anthem. The exact key of the opening anthem is a well-kept secret. However, there are some rumours. Which of the following keys are ~~musically~~ geometrically sound?

$$\begin{array}{ll}
\mathrm{R}^\sharp = \mathrm{Rm} & \text{BL} \\
\mathrm{R}^\flat = \mathrm{Rm} & \text{JU} \\
\mathrm{Rm}^\sharp = \mathrm{R} & \text{PI} \\
\mathrm{Rm}^\flat = \mathrm{R} & \text{OR}
\end{array}$$

Problem 13.2 (CAT-prize)**.** Commander Blorx is awarded the prestiguous *CAT-prize for non-felines* for his contributions to inter-planetary roguery. During the ceremony, Blorx is handed the ornate CAT cup, bearing his name. Unfortunately, the committee misspelled Blorx's name as $\mathrm{R}_{\mathrm{BLOX}}$. Which of the following inscriptions would be equivalent to $\mathrm{R}_{\mathrm{BLOX}}$?

$$\begin{array}{rl}
-\mathrm{R}_{\mathrm{BOLX}} & \text{XI} \\
\mathrm{R}_{\mathrm{BBOX}} & \text{TER} \\
\mathrm{R}_{\mathrm{OXBL}} & \text{TE} \\
-\mathrm{R}_{\mathrm{LBOX}} & \text{RI} \\
\mathrm{R}_{\mathrm{LOXB}} & \text{XIS} \\
-\mathrm{R}_{\mathrm{LBXO}} & \text{R} \\
\mathrm{R}_{\mathrm{XOOX}} & \text{ST}
\end{array}$$

Problem 13.3 (disc-shaped planets)**.** After these preliminaries, the actual scientific part of the congress begins. The first session is about disc-shaped planets (which used to be the state of the art for a long time!). Which of the following properties can a complete Riemannian metric have on an open disc in the plane?

$$\begin{array}{ll}
\sec < 0 & \text{SFL} \\
\sec = 0 & \text{ATT} \\
\sec > 0 & \text{NOT}
\end{array}$$

Please turn over

Problem 13.4 (planets of constant curvature). Currently it is fashionable to create planets of constant sectional curvature. Which of the following smooth manifolds admit a Riemannian metric of constant sectional curvature?

$\mathbb{R}^2 \times \mathbb{R}^2$	ER
$\mathbb{S}^2 \times \mathbb{R}^2$	APL
$\mathbb{S}^2 \times \mathbb{S}^2$	A

Problem 13.5 (doubly hyperbolic cookies). The congress then meanders into over-specialised areas. While others attend the talks, Blorx manages to find the cookie stash. The cookies are doubly hyperbolic, i.e., cookies that look like $\mathbb{H}^2 \times \mathbb{H}^2$ with the product Riemannian metric of the hyperbolic Riemannian metrics of sectional curvature -1. Which properties do such cookies have?

$\sec < 0$	NET
$\text{scal} < 0$	TH
constant scalar curvature	AN

Problem 13.6 (mixed-type planets). The top 1% indulge life on mixed-type planets diffeomorphic to $\mathbb{H}^2 \times \mathbb{S}^2$. Which of the following luxurious properties can a complete Riemannian metric on such a planet have?

flatness	NOR
$\sec > 0$	AMO
$\text{scal} > 0$	EAR

Problem 13.7 (excursion). The excursion consists of a visit to the *Metroplanetary Museum of Art*. Which famous planet design principle is depicted in the painting below?

Cartan-Hadamard	ON
Bonnet-Myers	S
Hopf-Rhino	TH

Blorx takes advantage of the excursion to slip away from the congress and to start his well-deserved break ...

Solution self-check:

No submission!

C

General information

Differential Geometry I: Admin

Prof. Dr. C. Löh/AG Ammann October 2020

Homepage. Information and news concerning the lectures, exercise classes, office hours, literature, as well as the weekly assignments can be found on the course homepage and in GRIPS:

http://www.mathematik.uni-regensburg.de/loeh/teaching/diffgeo_ws2021

https://elearning.uni-regensburg.de/course/view.php?id=45748

Virtual teaching. In view of the COVID-19 pandemic, until further notice, this course will be taught remotely, based on:

- **Guided self-study.** There will be reading assignments for each lecture (except the first), based on extensive lecture notes.
- **Remote lecture sessions.** During the time slots of the lectures, there is the opportunity to discuss the topics of these reading assignments and ask questions in a video conference.
- **Remote exercise sessions.** Solutions to the exercise sheets are to be submitted by email to your tutor. These submissions will be graded and returned digitally.

  Questions/solutions will be discussed in a video conference during the time slots of the exercise classes.
- **Text chat.** In addition, there is also an informal chat forum for discussing questions or sharing virtual meeting coordinates with fellow participants (the invite link is provided in GRIPS).

Video conferences. For video conferences, we will probably use zoom (uni-regensburg.zoom.us). Access information will only be provided through GRIPS (and not publicly on the homepage). Some guidelines for these video conferences:

- Even though we are not sitting in the same room, it might help to imagine that we do.
- Be on time.
- Do *not* record these meetings. Not even partially.
- Focus your attention on the discussion (and do not try to do zillions of other things at the same time ...).
- Mute your microphone while you are not talking (this can improve overall audio quality significantly).
- If bandwidth and the living situation allows for video, it would be very much appreciated if you could switch on your camera – in this way, for everybody, interaction will be more immediate and more natural.
- Feel free to use the zoom chat to talk to other participants during the lectures (on lecture related topics ...). However, usually, I will not be able to monitor questions on the chat during the lectures.

- It might be a good idea to familiarise yourself with zoom and your hardware before the first meeting.

In case you do not have sufficiently powerful hardware or a sufficiently strong internet connection, this should not be an issue: I will make an effort of integrating all essential information into the lecture notes and there is a GRIPS forum in which you can ask questions.

Lectures. The lectures will take place on Wednesdays (8:30–10:00) and on Thursdays (10:15–12:00). The first lecture will be on Wednesday, November 4, at 8:30. The lecture notes can be found on the course homepage and in GRIPS. The last lecture will be on February 11.

Why are there no recordings of the lectures? During the COVID-19 pandemic, the lectures provide precious interaction time. I want to encourage all participants to actively engage in discussions and both asking and answering questions. Therefore, I want to keep the atmosphere as casual, non-formal, and non-intimidating as possible.

Exercises. Homework problems will be posted on Thursdays (before 10:00) on the course homepage/GRIPS; submission is due one week later (before 10:00) by email to your tutor.

Each exercise sheet contains four regular exercises (4 credits each) and more challenging bonus problems (4 credits each).

It is recommended to solve the exercises in small groups; however, solutions need to be written up individually (otherwise, no credits will be awarded). Solutions can be submitted alone or in teams of at most two participants; all participants must be able to present *all* solutions of their team.

The exercise classes start in the *second* week; in this first session, some basics on manifolds will be discussed.

Quick checks. In addition, the lecture notes contain quick checks that will train elementary techniques and terminology. These problems should ideally be easy enough to be solved within a few minutes. Solutions are not to be submitted and will not be graded. These quick checks will have feedback implemented directly in the PDF.

Moreover, the weekly assignment sheets will also contain simple computational exercises (which also will not be submitted/graded). These will give the opportunity to practise basic computational techniques.

Registration for the course. Please register for the course via GRIPS:

https://elearning.uni-regensburg.de/course/view.php?id=45748

There will be two groups of exercise classes:

- Monday, 8:00–10:00
- Tuesday, 8:00–10:00

Registration deadline for the exercise classes: Thursday, November 5, 14:00. We will try to fill the groups according to your preferences. The distribution will be announced on Friday, November 6.

Credits/Exam. This course can be used as specified in the commented list of courses and in the module catalogue.

- *Studienleistung*: Successful participation in the exercise classes: 50% of the credits (of the regular exercises), presentation of a solution in class, active participation (the last two items will be interpreted appropriately during virtual teaching).
- *Prüfungsleistung*: Oral exam (25 minutes), by individual appointment at the end of the lecture period/during the break. As of now, the default will be an online oral exam.

You will have to register in FlexNow for the Studienleistung and the Prüfungsleistung (if applicable).

Further information on formalities can be found at:

http://www.uni-regensburg.de/mathematik/fakultaet/studium/studierende-und-studienanfaenger/index.html

Contact.

- If you have questions regarding the organisation of the exercise classes, please contact Matthias Ludewig:

  matthias.ludewig@ur.de

- If you have questions regarding the exercises, please contact your tutor.
- If you have mathematical questions regarding the lectures, please contact your tutor or Clara Löh.
- If you have questions concerning your curriculum or the examination regulations, please contact the student counselling offices or the exam office:

http://www.uni-regensburg.de/mathematik/fakultaet/studium/ansprechpersonen/index.html

- Official information of the administration related to the COVID-19 pandemic can be found at:

  https://go.ur.de/corona

In many cases, also the Fachschaft can help:

http://www-cgi.uni-regensburg.de/Studentisches/FS_MathePhysik/cmsms/

Differential Geometry I: More Literature

Prof. Dr. C. Löh/AG Ammann 30.10.2020

In case the COVID-19 pandemic leaves you stranded at home with too much time for reading, here are some suggestions connected to differential geometry:

- E. A. Abbott. *Flatland, A Romance of Many Dimensions*, Dover Publications, 1992.
- E. Brooke-Hitching. *The Phantom Atlas: The Greatest Myths, Lies and Blunders on Maps*, Simon+Schuster, 2016.
- W. Chyr. *Manifold Garden*, computer game, 2019.
- R. Coulon, E. A. Matsumoto, H. Segerman, S. J. Trettel. Ray-marching Thurston geometries, preprint, arXiv:2010.15801 [math.GT], 2020.
- M. C. Escher. *M. C. Escher – The official website*, http://www.mcescher.com/
- M. Gessen. *Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century*, Houghton Mifflin Harcourt, 2009.
- Mobius Digital. *Outer Wilds*, computer game, 2019.
- Star Trek. *The Next Generation: The Vengeance Factor*, episode 3x09, 1989.
  See also: https://memory-alpha.fandom.com/wiki/Riemannian_tensor_field
- Valve. *Portal*, computer game, 2007.

Bibliography

Please note that the bibliography will grow during the semester. Thus, also the numbers of the references will change!

[1] B. Ammann. *Analysis auf Mannigfaltigkeiten*, lecture notes, summer term 2020, Universität Regensburg.
http://www.mathematik.uni-regensburg.de/ammann/lehre/2020s_analysisIV/AnalysisIV-V2020-10-26.pdf
Cited on page: 16, 17, 19, 20, 71, 74, A.9

[2] M. F. Atiyah. *K-Theory*, notes by D. W. Anderson, second edition, Advanced Book Classics, Addison-Wesley Publishing Company, 1989. Cited on page: 37

[3] C. Bär. *Differential Geometry*, lecture notes, summer term 2013, Universität Potsdam.
https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/skript-DiffGeo-engl.pdf Cited on page: 45

[4] S.-M. Belcastro, C. Yackel (eds). *Making mathematics with needlework*, ten papers and ten projects, A K Peters, 2008. Cited on page: 56

[5] M. R. Bridson, A. Haefliger. *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, 319, Springer, 1999. Cited on page: 2, 181, 190

[6] J. Cheeger, D. G. Ebin. *Comparison theorems in Riemannian geometry*, revised reprint of the 1975 original, AMS, 2008. Cited on page: 180

[7] M.W. Davis. *The Geometry and Topology of Coxeter Groups*, London Mathematical Society Monographs, 32, Princeton University Press, 2008. Cited on page: 182

[8] C. Druţu, M. Kapovich. *Geometric Group Theory*, with an appendix by Bogdan Nica, volume 63 of *Colloquium Publications*, AMS, 2018. Cited on page: 2, 181, 189, 190, 191

[9] M.C. Escher. *Cirkellimiet IV*, 1960. Available online at http://www.mcescher.nl/galerij/erkenning-succes/cirkellimiet-iv/ Cited on page: 56

[10] M. Gessen. *Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century*, Houghton Mifflin Harcourt, 2009. Cited on page: 4

[11] M. Gromov. Groups of polynomial growth and expanding maps, with an Appendix by Jacques Tits, *Publ. Math. Inst. Hautes Études Sci.*, 53, pp. 53–78, 1981. Cited on page: 191

[12] M. Gromov. Hyperbolic groups, Essays in group theory, *Math. Sci. Res. Inst. Publ.*, 8, pp. 75–263, Springer, 1987. Cited on page: 2

[13] R. .S. Hamilton. Three-manifolds with positive Ricci curvature, *J. Differential Geometry*, 17(2), pp. 255–306, 1982. Cited on page: 3

[14] P. de la Harpe. *Topics in Geometric Group Theory*, Chicago University Press, 2000. Cited on page: 2

[15] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, corrected reprint of the 1978 original. Graduate Studies in Mathematics, 34. American Mathematical Society, 2001. Cited on page: 48, 61, 134, 149

[16] B. Kleiner, J. Lott. Notes on Perelman's papers, *Geom. Topol.*, 12(5), pp. 2587–2855, 2008. Cited on page: 3

[17] J. M. Lee. *Introduction to smooth manifolds*, Graduate Texts in Mathematics, 218, Springer, 2003. Cited on page: 25

[18] J. M. Lee. *Introduction to Riemannian manifolds*, second edition, Graduate Texts in Mathematics, 176, Springer, 2018. Cited on page: 55, 64, 98, 118, 134, 136, 149, 172

[19] C. Löh. *Geometric Group Theory. An Introduction*, Universitext, Springer, 2018. Cited on page: 2, 181, 189, 190, 191

[20] C. Löh. *Group Cohomology* lecture notes, Universität Regensburg, 2019. Cited on page: 182, 188

[21] J. Matoušek. *Using the Borsuk-Ulam Theorem*, Lectures on topological methods in combinatorics and geometry. Written in cooperation with Anders Björner and Günter M. Ziegler, *Universitext*, Springer, 2003. Cited on page: 60

[22] J. Milnor. On manifolds homeomorphic to the 7-sphere, *Ann. of Math. (2)*, 64, pp. 399–405, 1956. Cited on page: 20

[23] J. Milnor. A note on curvature and the fundamental group, *Journal of Differential Geometry*, 2, pp. 1–7, 1968. Cited on page: 190

[24] J. W. Milnor, J. D. Stasheff. *Characteristic classes*, Annals of Mathematics Studies, 76, Princeton University Press, University of Tokyo Press, 1974. Cited on page: 30

[25] E. E. Moise. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung, *Ann. of Math. (2)*, 56, pp. 96–114, 1952. Cited on page: 20

[26] J. Morgan, G. Tian. *The geometrization conjecture*, Clay Mathematics Monographs, 5, American Mathematical Society, Clay Mathematics Institute, 2014. Cited on page: 3

[27] G. Perelman. The entropy formula for the Ricci flow and its geometric applications, preprint, arXiv:math/0211159, 2002. Cited on page: 3

[28] G. Perelman. Ricci flow with surgery on three-manifolds, preprint, arXiv:math/0303109, 2003. Cited on page: 3

[29] G. Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, preprint, arXiv:math/0307245, 2003. Cited on page: 3

[30] J. G. Ratcliffe. *Foundations of hyperbolic manifolds*, second edition, Graduate Texts in Mathematics, 149, Springer, 2006. Cited on page: 55

[31] C. A. Weibel. *The K-Book. An Introduction to Algebraic K-Theory*, Graduate Studies in Mathematics, 145. American Mathematical Society, 2013. Cited on page: 37

[32] Zeno Rogue. *HyperRogue*, computer game, https://github.com/zenorogue/hyperrogue Cited on page: 56

Deutsch → English

A

Atlas	atlas	10
Automorphismengruppe	automorphism group	A.11

B

Bündelabbildung	bundle map	23
Bahn	orbit	A.12
Bahnenraum	orbit space	A.12
Beschleunigung	acceleration	90
Blatt	sheet	58
Bogenlänge	arc-length	64

D

Derivation	derivation	17
Diffeomorphismus	diffeomorphism	15
Differentialgeometrie	differential geometry	1

E

eigentliche Operation	proper action	57
Einsteinsche Summenkonvention	Einstein summation convention	12
erste Variationsformel	first variation formula	140

F

Faser	fibre	22
flach	flat	115
Fluss	flow	3
freie Gruppenoperation	free action	A.12

Fundamentalform	fundamental form	42
Funktor	functor	A.5

G

Geodätische, geodätisch	geodesic	70
Geometrisierung	geometrisation	4
Geradenbündel	line bundle	22
glatt	smooth	9
Gruppenoperation	group action	A.11

H

Hauptachsentransformation	principal axis theorem	183
homogen	homogeneous	46
hyperbolischer Raum	hyperbolic space	50

I

Identitätsmorphismus	identity morphism	A.3
Isometrie	isometry	43
Isomorphismus	isomorphism	A.4
isotrop	isotropic	46

J

Jacobi-Feld	Jacobi field	158

K

Karte	chart	9
Kategorie	category	A.3
konjugiert	conjugate	167
konjugierter Punkt	conjugate point	166
kontravarianter Funktor	contravariant functor	A.5
kovariante Ableitung	covariant derivative	81
kovarianter Funktor	covariant functor	A.5
Krümmung	curvature	77

M

Mannigfaltigkeit	manifold	8
Morphismus	morphism	A.3

O

Objekt	object	A.3
Orbit	orbit	A.12

P

Parallelenaxiom	parallel postulate	157
Paralleltransport	parallel transport	92
Parametrisierung	parametrisation	11
Partition der Eins	partition of unity	A.9

English → Deutsch

A

acceleration	Beschleunigung	90
arc-length	Bogenlänge	64
atlas	Atlas	10
automorphism group	Automorphismengruppe	A.11

B

bundle map	Bündelabbildung	23

C

category	Kategorie	A.3
chart	Karte	9
conjugate	konjugiert	167
conjugate point	konjugierter Punkt	166
connection	Zusammenhang	80
contravariant functor	kontravarianter Funktor	A.5
covariant derivative	kovariante Ableitung	81
covariant functor	kovarianter Funktor	A.5
cover	Überdeckung	58
covering	Überlagerung	58
curvature	Krümmung	77

D

derivation	Derivation	17
diffeomorphism	Diffeomorphismus	15
differential geometry	Differentialgeometrie	1

E

Symbols

Index

D

E

F

Q

R

W

Don't click me!