02.12.2020

Recap: model spaces: S^n, $\mathbb{R}^n$, $\mathbb{H}^n$ and their symmetries

II.4.5 Group Actions

<u>idea:</u> sometimes it is good to divide out (some) symmetries

<u>but:</u> usually: actions $\Gamma \curvearrowright M$ lead to "bad" quotient spaces (unless the action is "nice").

<u>here:</u> consider actions that are free and proper.

DE: eigentlich

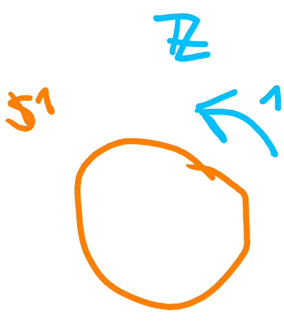
Definition. (proper action). A continuous action $\Gamma \curvearrowright M$ of a (discrete) group Γ on a top. space M is <u>proper</u> iff for every compact set $K \subset M$, the set

$$\{\gamma \in \Gamma \mid \gamma \cdot K \cap K \neq \emptyset\} \subset \Gamma$$

is finite.

(Non-) Examples:

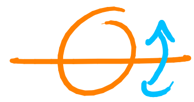
- translation action $\mathbb{Z} \curvearrowright \mathbb{R}$: proper
- translation action $\mathbb{Q} \curvearrowright \mathbb{R}$: <u>not</u> proper
- on compact spaces: actions are proper iff and only if the acting group is finite.



In particular: Irrational rotations $\mathbb{Z} \curvearrowright S^1$ are free but <u>not</u> proper. irrat rot $\backslash S^1$ is <u>not</u> Hausdorff

Warning: free $\overset{\text{i.g.}}{\not\Rightarrow}$ proper ← e.g. irrat. rot. $\mathbb{Z} \curvearrowright S^1$

proper $\underset{\text{i.g.}}{\not\Rightarrow}$ free ← e.g. $\mathbb{Z}/2 \curvearrowright S^1$



Remark (free + proper). Let $\Gamma \curvearrowright M$ be a free and proper action of a group on a top. mfd M.

- Then $\Gamma \curvearrowright M$ is <u>properly discontinuous</u>: For every $x \in M$, there ex. an open nbhd $U \subset M$ of x s.t.

$$\forall_{\gamma \in \Gamma \setminus \{e\}} \quad \gamma \cdot U \cap U = \emptyset.$$

- Then: the quotient map $p : M \longrightarrow \Gamma \backslash M$ (quotient top.) is a <u>covering map</u> (DE: Überlagerung): For every $x \in \Gamma\backslash M$, there ex. an open nbhd $U \subset \Gamma \backslash M$ of x and a homeo $f : p^{-1}(U) \longrightarrow U \times \Gamma$ (discrete top.) s.t.

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "f"] \arrow[d, "p|"'] \arrow[dr, phantom, "\text{Top}"] & U \times \Gamma \arrow[d, "\text{proj. to } U"] \\ U \arrow[r, equal] & U \end{tikzcd}$$

⇝ p is an open map and a local homeo

$U \times \{\gamma\}$: "sheets" over U

Moreover: we can choose U and f in such a way that f is Γ-equivariant (where $\Gamma \curvearrowright U \times \Gamma$ via left translation $\Gamma \curvearrowright \Gamma$).

Now: how to get a smooth structure
• Riem. metric on the quotient?

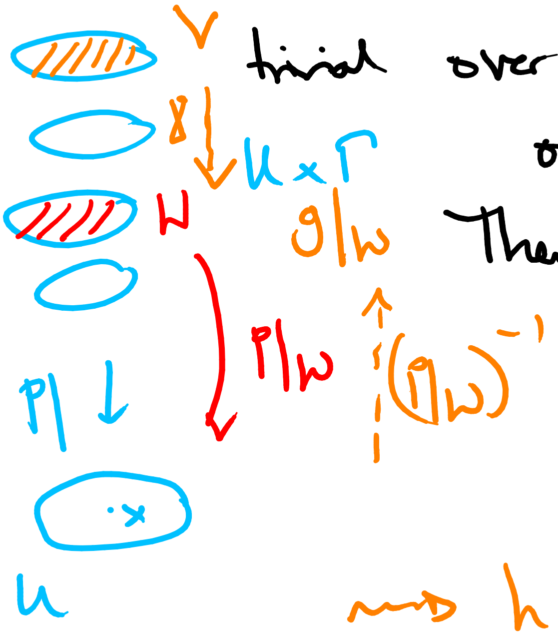
Proposition. (smooth structure on quotients). Let $\Gamma \curvearrowright M$ be a free and proper action of a group Γ on a smooth manifold M by diffeos.
Then: The quotient top. space $\Gamma \backslash M$ admits a unique smooth structure that turns the can. proj. $M \to \Gamma \backslash M$ into a local diffeo.

Proof. idea: The proj. $M \to \Gamma\backslash M$ is a covering map. We then use local triviality to define the smooth structure on $\Gamma \backslash M$ "locally". $\square$

Proposition. (Riemannian metrics on quotients). Let (M,g) be a Riem mfd and let $\Gamma \curvearrowright (M,g)$ be a free and proper action of a group Γ by isometries of (M,g). Then the smooth mfd $\Gamma\backslash M$ admits a unique Riemannian metric that turns the can. proj. $p: M \to \Gamma \backslash M$ into a local isometry.

previous prop.

Proof. We know that $p: M \to \Gamma \backslash M$ is a covering map and a local diffeo.
• Locally: let $x \in \Gamma \backslash M$ and let $U \subset \Gamma \backslash M$ be an open nbhd of x s.t. p is Γ-equiv.



trivial over U. Let $W \subset p^{-1}(U)$ be one of the sheets over U.

Then: $p|_W : W \to U$ is a diffeo.

Let
$$h_U := \left((p|_W)^{-1}\right)^*(g|_W).$$

$\rightsquigarrow$ h_U is a Riem. metric on U.

Moreover: h_U is indep. of the sheet W: let $V \subset p^{-1}(U)$ also be a sheet. Then, there ex. $\gamma \in \Gamma$ with $p|_V = p|_W \circ (\gamma \cdot \cdot)$.

Because $\gamma \cdot \cdot : M \to M$ is an isometry, we obtain $\left((p|_V)^{-1}\right)^*(g|_V) = \left((p|_W)^{-1}\right)^*(g|_W)$.

- Globally: glue these local constructions.
- Uniqueness: follows from p being a local isometry. $\square$

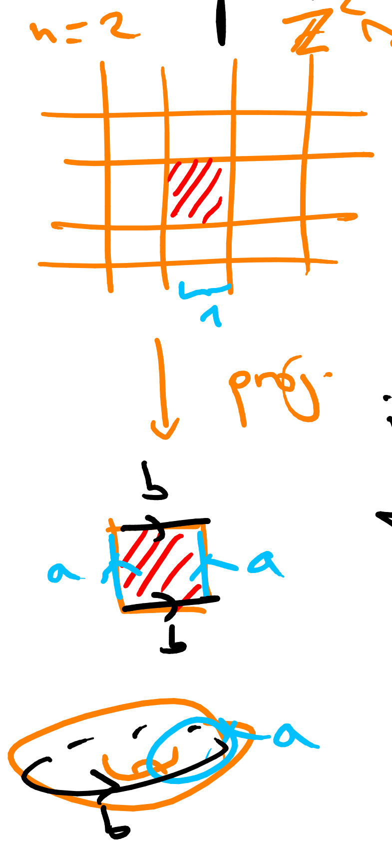
Example. (real projective spaces). Let $n \in \mathbb{N}$. Then:

$$\mathbb{R}P^n := \mathbb{Z}/2 \backslash S^n$$

antipodal action:
$\mathbb{Z}/2 \times S^n \to S^n$
$([1], x) \mapsto -x$

$\rightsquigarrow$ $\mathbb{R}P^n$ inherits a (free and proper!) smooth structure from the standard smooth structure on S^n and a Riem. metric from the round metric on S^n.

Example (tori). Let $n \in \mathbb{N}$. Then



$\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ translation action $\mathbb{Z}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(x,y) \mapsto x+y$

is isometric to $\left(S^1\left(\frac{1}{2\pi}\right)\right)^{\times n}$ with the product of the round metric.

1.5 Towards Riemannian Geometry

idea: Riemannian metric $\rightsquigarrow$ distances, (angles), volumes, ...

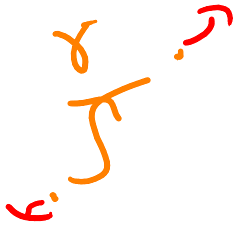
How to define distances on Riem. mflds?

Strategy:

- define the length of curves as the integral of the speed (length of the velocity) of the curve. — measured wrt the Riem. metric on the tangent spaces
- define the distance between two points as the inf over all lengths of curves connecting these points.

1.5.1 Length of curves

Definition ((piecewise) smooth/regular curves). Let M be a smooth mfd, let $a,b \in \mathbb{R}$ with $a<b$.



- A <u>smooth curve</u> $[a,b] \to M$ is a map $[a,b] \to M$ that admits a smooth extension to an open interval that contains $[a,b]$.

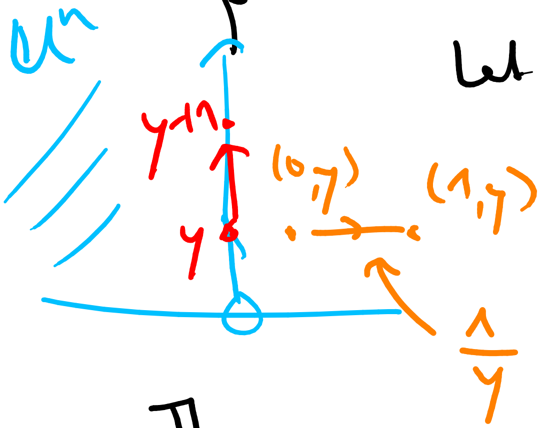
  A smooth curve is <u>regular</u> if $d_t\gamma \neq 0$ for all $t \in [a,b]$.

- A <u>piecewise regular smooth curve</u> $[a,b] \to M$ is a map $\gamma : [a,b] \to M$ s.t. there ex. $k \in \mathbb{N}$ and $a = a_0 < a_1 < \dots < a_k = b$ s.t.
$$\gamma|_{[a_j, a_{j+1}]} \text{ is regular smooth for each } j \in \{0, \dots, k-1\}.$$

Definition (length of a curve). Let (M,g) be a Riem. mfd and let $\gamma : [a,b] \to M$ be a piecewise smooth curve. Then, the <u>length of γ</u> wrt g is

$$L_g(\gamma) := \int_a^b \underbrace{\big\| \overbrace{\dot\gamma(t)}^{= d_t\gamma(1) \in T_{\gamma(t)}M} \big\|_{g_{\gamma(t)}}}_{\text{integrable function (!)}} \, dt \in \mathbb{R}_{\geq 0}.$$

Example in $\mathbb{U}^n(1)$:



Let $y \in \mathbb{R}_{>0}$. and let

$$\gamma_y : [0,1] \to \mathbb{U}^n(1)$$
$$t \mapsto (0, y+t).$$

Then

$$L_{g^{\mathbb{U}}}(\gamma_y) = \int_0^1 \|\dot\gamma_y(t)\|_{g^{\mathbb{U}}_{\gamma(t)}} \, dt$$

$$= \int_0^1 \frac{1}{y+t} \cdot \underbrace{\|(0,1)\|_2}_{=1} \, dt$$

$$= \int_0^1 \frac{1}{y+t} \, dt = \ln(y+1) - \ln y.$$