10.02.2021

Recap: Goal: · global (top.) consequences of curvature constraints such as $\sec < 0$, $\sec \leq 0$, $\sec > 0$, ...

· more on constant sectional curvature

$\boxed{4.3}$ ANALYTIC & GEOMETRIC COMPARISON THEOREMS

alternatively: Riccati eq.

Analytically: upper bounds on sec
$\rightsquigarrow$ lower bounds for Jacobi fields

Geometrically: $\rightsquigarrow$ estimates for conjugate points and comparison results for Riem. metrics

Theorem (Sturm comparison theorem). Let $T \in \mathbb{R}_{>0}$, let $a : [0,T) \to \mathbb{R}$ be a function. Let $u, v : [0,T) \to \mathbb{R}$ be differentiable, and twice differentiable on $(0,T)$ with $u|_{(0,T)} > 0$ and

$$u'' + a \cdot u = 0 \qquad u(0) = 0 = v(0)$$

$$v'' + a \cdot v \geq 0 \qquad u'(0) = v'(0) > 0.$$

Then

$$\forall_{t \in [0,T)} \quad v(t) \geq u(t).$$

Proof. Idea: consider $[0,T) \to \mathbb{R}$

$$t \mapsto \begin{cases} \frac{v(t)}{u(t)} & \text{if } t>0 \\ 1 & \text{if } t=0. \end{cases}$$

+ double differentiation.

+ some technicalities. $\square$

Theorem. (Jacobi field comparison theorem).

Let (M,g) be a Riem mfd, let $c \in \mathbb{R}$, and let $\boxed{\sec \leq c}$. Let $\gamma: I \to M$ be a unit speed geodesic with $0 \in I$ and let V be a normal Jacobi field along γ with $V(0)=0$.

Then:

$$\forall_{t \in I} \quad \|V(t)\|_g \geq \begin{cases} \|D_\gamma V(0)\|_g \cdot t & \text{if } \boxed{c=0}, t>0 \\ \|D_\gamma V(0)\|_g \cdot R \cdot \sinh \frac{t}{R} & \text{if } \boxed{c<0}, t>0 \\ \|D_\gamma V(0)\|_g \cdot R \cdot \sin \frac{t}{R} & \text{if } \boxed{c>0}, \\ & t \in I \cap]0, \pi R] \end{cases}$$

where $R := \frac{1}{\sqrt{|c|}}$

Proof. Idea: Apply the Sturm comparison thm on $[0,T_c)$ with $T_c := \begin{cases} \infty & \text{if } c \leq 0 \\ \pi \cdot R & \text{if } c>0 \end{cases}$ to the functions

- u: gen. sine function for c
- $v := \|V\|_g = \sqrt{\langle V,V \rangle_g}$

(technical problem: v is smooth at all $t \in I$ with $V(t) \neq 0$; but one has to deal also with t with $V(t) = 0$.)

- Key estimate: v satisfies $v'' + c \cdot v \geq 0$.

Let $t \in I$ with $V(t) \neq 0$. Then

$$v' = \|V\|_g' = \left(\sqrt{\langle V, V\rangle_g}\right)' = \cancel{\tfrac{1}{2}}\cdot \cancel{2} \cdot \frac{\langle D_\gamma V, V\rangle_g}{\langle V, V\rangle_g^{1/2}}.$$

$$v'' = \frac{1}{\langle V, V\rangle_g} \cdot \Big(\langle D_\gamma D_\gamma V, V\rangle_g \cdot \langle V, V\rangle_g^{1/2} + \langle D_\gamma V, D_\gamma V\rangle_g \cdot \langle V, V\rangle_g^{1/2}$$
$$- \langle D_\gamma V, V\rangle_g \cdot \langle V, V\rangle_g^{-1/2} \cdot \langle D_\gamma V, V\rangle_g \Big)$$

Jacobi eq. !

$$= \frac{\langle -R(V, \dot\gamma, \dot\gamma), V\rangle_g}{\|V\|_g} + \underbrace{\frac{\|D_\gamma V\|_g^2 \cdot \|V\|_g^2 - \langle D_\gamma V, V\rangle_g^2}{\|V\|_g^3}}_{\geq 0 \quad (CS!)}$$

$$\geq - \frac{\langle R(V, \dot\gamma, \dot\gamma), V\rangle_g}{\|V\|_g}$$

$$= - \frac{Rm(V, \dot\gamma, \dot\gamma, V)}{\|V\|_g}$$

$$\rightsquigarrow \boxed{v''(t)} \geq - \sec_{\gamma(t)}(V(t), \dot\gamma(t)) \cdot \|V(t)\|_g$$

$$\boxed{\geq - c \cdot v(t).}$$

$\square$

see $\sec(v, w) = \dfrac{Rm(v, w, w, v)}{\text{area}(v, w)}$

Corollary. (conj. pt comparison thm). Let (M,g) be a Riem. mfd, let $c \in \mathbb{R}$, and $\sec \leq c$.

1. If $c \leq 0$, then M has <u>no</u> conjugate points along any geodesic.
2. If $c > 0$, then the first conj. pt along any geodesic in M cannot occur before a distance of $\pi \cdot R$, where $R = \frac{1}{\sqrt{c}}$.

$\square$

Corollary. (Riem. metric comparison thm). Let (M,g) be a Riem. mfd, let $c \in \mathbb{R}$, and $\sec \leq c$. Let U be a normal nbhd centered at $x \in M$, let $y \in U$, and $v \in T_y M$. If $c > 0$, then we also require $d_g(x,y) \leq \pi \cdot \frac{1}{\sqrt{c}}$. Then

$$\|v\|_g \geq \|v\|_{g_c}$$

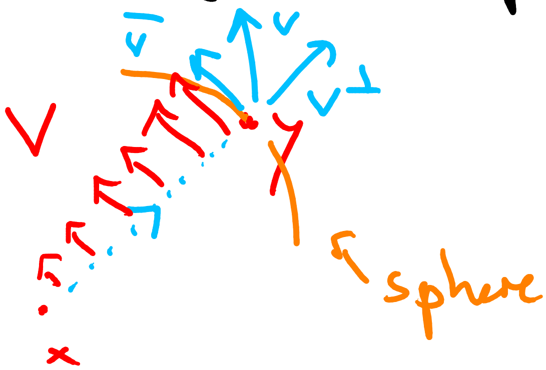
as in Thm 4.2.4

constant sectional curv. c

Idea of proof: Use the Gauss lemma to decompose v.

- for the radial contribution: clear.

· for the contribution tangent to the geodesic sphere around x through y:



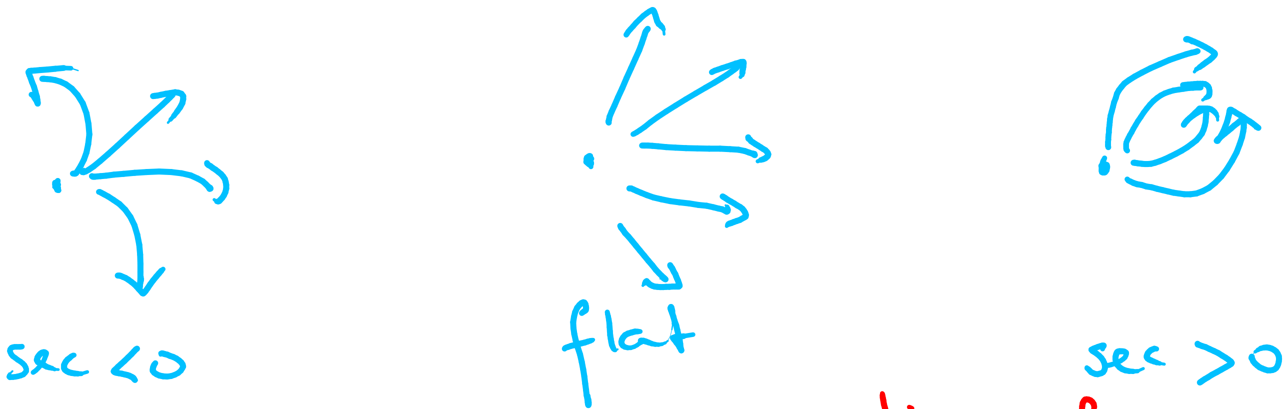
· take a suitable Jacobi field along radial geodesic from x to y

· apply the Jacobi field comp. thm.

· computation + comparison with the def of g_c. □

⟿ more geometric applications...

· How do geodesics spread?



notions of curvature without tensors

· comparison results for distances

⟿ comparison geometry (CAT(κ)-spaces, δ-hyperbolic spaces).

4.4 NON-POSITIVE CURVATURE

Hopf-Rinow!

Theorem. (Cartan-Hadamard theorem).
Let (M,g) be a complete Riem mfd with $M \neq \emptyset$ with $\boxed{\sec \leq 0}$.

1. If $x \in M$, then $\mathrm{Exp}_x = T_xM$ and $\exp_x : T_xM \to M$ is a covering map.
2. In particular: "The" universal covering space of M is diffeomorphic to $\mathbb{R}^{\dim M}$. If M is simply connected, then $M \underset{\text{diffeo}}{\cong} \mathbb{R}^{\dim M}$.

Proof. Let $n := \dim M$.

1. • By completeness: $\mathrm{Exp}_x = T_xM$.

• $\sec \leq 0$

$\Downarrow$ conj. pt. comp. thm

no conjugate pts on M

$\Downarrow$ Prop. 3.5.14

$\exp_x$ is a local diffeo

• Let $\tilde{g} := \exp_x^* g$ (Riem. metric on T_xM).

By constr.: $\exp_x$ is a local isometry wrt $\tilde{g}$ and g.

- Then $(T_xM, \tilde{g})$ is complete (Ü 11...) and thus $\exp_x : T_xM \to M$ is a covering map (Ü 11.8).

[1. ⇒ 2.: covering theory] □

Corollary. Let (M,g) be a simply connected complete Riem. mfd and let $c \in \mathbb{R}_{<0}$ with $\boxed{\sec \leq c < 0.}$ Let $x \in M$. Then

$$\beta_x^{M,g} : r \longmapsto \mathrm{vol}_M\left(B_r^{d_g}(x)\right)$$

grows at least exponentially.

Proof. By Cartan–Hadamard: M is a normed nbhd of x.

⇝ Apply the Riem metric comp. thm to M and $\underbrace{\mathbb{H}^n(\cdot\sim)}$.
+ Ü 12.4

vol growth is at least exponential (Ü 6.2). □

Example (for CHT): Let $n \geq 2$. Then S^n and $\mathbb{RP}^n$ do <u>not</u> admit any Riem. metric of non-pos. sectional curvature.

(the universal covering is S^n, which is <u>not</u> homeomorphic to $\mathbb{R}^n$.)

Theorem. Let (M,g) be a complete simply connected Riem. mfd with constant sectional curvature c and let $n := \dim M$. $M \neq \emptyset$

1. If $c = 0$, then (M,g) is isometric to $\mathbb{R}^n$ with the Euclidean Riem. metric.
2. If $c < 0$, then (M,g) is isometric to $\mathbb{H}^n(R)$ with $R = \frac{1}{\sqrt{|c|}}$.

Proof. Apply Cartan-Hadamard and the description of constant sec. curvature metrics in normal nbhds (Thm 4.2.4) + knowledge on sec of model spaces. $\square$