14.01.2021

Recap: · $R \rightsquigarrow Rm = R^\flat$

· characterisation of flat mfds

Goal: extract/forget information of the Riem. curvature tensors in a convenient way.

Idea: Rm $\rightsquigarrow$ sec — sectional curvature
using the symmetries well

R

$\downarrow$ average

$\rightsquigarrow$ tr — Ric — Ricci curvature tensor

tr $\downarrow$ average

scal — scalar curvature

2.4.3 SECTIONAL CURVATURE

DE: Schnittkrümmung

Idea: sectional curvature: tangent plane $\longmapsto$ real number
spanned by v, w

$\downarrow$

$Rm(v,w,w,v)$ scaled appropriately

Definition (sectional curvature). Let (M,g) be a Riem. mfd, let $x \in M$. Then the <u>sectional curvature of (M,g) at x</u> is def'd by: For all li. indep. pairs (v,w) with $v,w \in T_xM$:

$$\sec_x^{M,g}(v,w) := \frac{Rm(v,w,w,v)}{|v \wedge w|_g^2}$$

$$\underbrace{|v \wedge w|_g := \sqrt{\underbrace{\|v\|_g^2 \cdot \|w\|_g^2 - \langle v, w\rangle_g^2}_{\geq 0 \text{ CS}}}}_{\text{area of } w \,/\!/\!/\!/\, v}$$

Example. Flat Riem. mfds have constant sectional curvature 0.

Remark. If $\varphi : (M_1, g_1) \to (M_2, g_2)$ is a local isometry and $x \in M_1$, $v, w \in T_x M_1$, lin. indep., then

$$\sec_x^{M_1, g_1}(v, w) = \sec_{\varphi(x)}^{M_2, g_2}(d_x\varphi(v), d_x\varphi(w))$$

(Rm and $|\cdot \wedge \cdot|_g$ are preserved by φ).

Proposition. Let (M, g) be a Riem. mfd, let $x \in M$, and let (v, w), $(\tilde{v}, \tilde{w})$ be two lin. indep. pairs in $T_x M$ with $\mathrm{span}_{\mathbb{R}}(v, w) = \mathrm{span}_{\mathbb{R}}(\tilde{v}, \tilde{w})$.

Then:
$$\sec(v, w) = \sec(\tilde{v}, \tilde{w}).$$

Proof. Because of $\mathrm{span}_{\mathbb{R}}(v, w) = \mathrm{span}_{\mathbb{R}}(\tilde{v}, \tilde{w})$, there ex. $A = \begin{pmatrix} \alpha_v & \alpha_w \\ \beta_v & \beta_w \end{pmatrix} \in GL(2, \mathbb{R})$ s.t.

$$\tilde{v} = \alpha_v \cdot v + \beta_v \cdot w \quad \text{and}$$
$$\tilde{w} = \alpha_w \cdot v + \beta_w \cdot w.$$

Then:

- $$|\tilde{v} \wedge \tilde{w}|_g^2 = \overbrace{|\det A|^2}^{= (\det A)^2} \cdot |v \wedge w|_g^2$$

- $\mathrm{Rm}(\tilde v, \tilde w, \tilde w, \tilde v) = \alpha_v\beta_w \cdot \alpha_v \beta_w \cdot \mathrm{Rm}(v,w,w,v)$
$+ \dots$ mixed terms

$(\det A)^2 \cdot \mathrm{Rm}(v,w,w,v)$

symmetry of Rm $\quad 0 = \{$ + terms with $\mathrm{Rm}(u,u,\cdot,\cdot)$ or $\mathrm{Rm}(\cdot,\cdot,u,u)$

$\square$

Proposition. (sectional curvature determines the Riem. curvature tensor). Let M be a smooth mfd and let R_1, R_2 be $(4,0)$-tensor fields on M that satisfy the symmetries in Prop. 2.4.9 of the Riem. curvature tensor. Moreover, for all $x \in M$ and all lin. indp $v, w \in T_xM$, we assume
$$R_1(v,w,w,v) = R_2(v,w,w,v).$$
Then: $R_1 = R_2$.

Proof. Look at $R_1 - R_2$ and use multilin. algebra (L) 9.3. $\square$

Corollary. (flatness via sectional curvature). A Riem. mfd is flat if and only if it has constant sectional curvature 0.

Proof. "$\Rightarrow$": Clear (above example).

"$\Leftarrow$": Apply the previous prop. to Rm and 0. $\square$

2.4.4 Ricci curvature

idea: $R \overset{\mathrm{tr}}{\rightsquigarrow} \mathrm{Ric}$

Definition. (Ricci curvature). Let (M,g) be a Riem. mfd. Then, the <u>Ricci (curvature) tensor</u> of (M,g) is the $(2,0)$-tensor field on M, def'd by

$$\mathrm{Ric} : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$$
$$(X,Y) \longmapsto \mathrm{tr}\big(Z \mapsto R(Z,X,Y)\big).$$

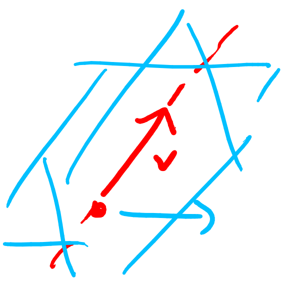
Example. Flat mfds have constant Ricci curvature tensor 0.

Remark. Let (M,g) be a Riem mfd. Then Ric can be recovered from the map

$$TM \longrightarrow \mathbb{R}$$
$$T_xM \ni v \longmapsto \mathrm{Ric}(v,v).$$

Ū 9.2.

Proposition. (Ricci via averaging of sec). Let (M,g) be a Riem mfd of dim n, let $x \in M$, let $v \in T_xM$, and let $(v_1, \dots, v_n)$ be an ONB of T_xM with $v_1 = v$.



Then

$$\mathrm{Ric}(v,v) = \sum_{k=2}^{n} \sec(v_k, v_{(1)}).$$

Proof. We have

$$\boxed{\mathrm{Ric}(v,v)} = \operatorname{tr}\big(z \mapsto R(z,v,v)\big)$$

use $(v_1,\dots,v_n)$ to unfold tr

$$= \sum_{i=1}^{n} v_i\text{-contribution of } R(v_i,v,v)$$

$(v_1,\dots,v_n)$ ONB

$$= \sum_{i=1}^{n} \langle R(v_i,v,v), v_i\rangle_g$$

$$= \sum_{i=\cancel{1}2}^{n} \mathrm{Rm}(v_i,v,v,v_i)\cdot \underbrace{\frac{1}{|v_i\wedge v|_g^2}}_{=1 \text{ as } (v_1,\dots,v_n) \text{ ONB}}$$

because $v_1 = v$ and symm of Rm

$$= \boxed{\sum_{i=2}^{n} \sec(v_i,v)}$$

$\square$

Remark. Ricci curvature is an invariant of local isometry.

2.4.5 Scalar curvature

idea: $\mathrm{Ric} \overset{\operatorname{tr}}{\rightsquigarrow} \mathrm{scal}$

Definition (scalar curvature). Let (M,g) be a Riem. mfd. Then, the <u>scalar curvature</u> of (M,g) is the smooth map

$$\begin{aligned}\mathrm{scal}\colon M &\longrightarrow \mathbb{R}\\ x &\longmapsto \operatorname{tr}\big(z \mapsto \mathrm{Ric}^{\#}_x(z)\big).\end{aligned}$$

Example. Flat mfds have constant scalar curv. 0.

Proposition (scal via averaging from sec). Let (M,g) be a Riem. mfd of dim n, let $x \in M$, and let $(v_1, \dots, v_n)$ be an ONB of $T_x M$. Then
$$\operatorname{scal}(x) = \sum_{\substack{j,k \in \{1,\dots,n\} \\ j \neq k}} \sec(v_j, v_k).$$

Proof. similar to the Ric-statement. $\square$

Remark. local isometries preserve scalar curvature.

Proposition. Homogeneous Riemannian mfds have constant scalar curvature.

Proof. Let (M,g) be a homogeneous Riem mfd and let $x, y \in M$. $\rightsquigarrow$ ex. $\varphi \in \operatorname{Isom}(M,g)$ with $\varphi(x) = y$.

inv. of scal
$$\rightsquigarrow \operatorname{scal}(x) = \operatorname{scal}(\varphi(x)) = \operatorname{scal}(y). \ \square$$

$\boxed{2.5}$ MODEL SPACES

$\underbrace{\mathbb{R}^n \checkmark, \ \mathbb{S}^n(R), \mathbb{H}^n(R)}_{\text{model spaces}}$

goal: compute sec of all model spaces

idea: use:
- high degree of symmetry $\rightsquigarrow$ constant sec. curvature
- locally conformally flat.

2.5.1 Locally conformally flat mfds

Definition.

- Let M be a smooth mfd. Riem. metrics $g, \tilde{g}$ are <u>conformal</u> if there ex. an $f \in C^\infty(M, \mathbb{R}_{>0})$ with
$$\tilde{g} = f \cdot g .$$
- A Riem. mfd (M,g) is <u>conformally flat</u> if g is conformal to a flat Riem. metric on M.
- A Riem. mfd (M,g) is <u>locally conformally flat</u>, if every $x \in M$ admits an open nbhd that is conformally flat (w.r.t. restriction of g).

Theorem. (Riem. curv. and conformal change)
Let (M,g) be a Riem. mfd of dim n, let $f \in C^\infty(M, \mathbb{R}_{>0})$, and let $\tilde{g} := e^{2f} \cdot g$.
Then: Locally, we have:

- $$\tilde{\Gamma}_{ij}^{k} = \boxed{\Gamma_{ij}^{k}} + \delta_{jk} \cdot E_i(f) + \delta_{ik} E_j(f) - g_{ij} \cdot E_k(f)$$
- $$\tilde{R}_{ijkl} = \boxed{e^{2f} \cdot R_{ijkl}}$$

$$- e^{2f} \cdot \left(g_{ik} \cdot T_{jl} + g_{jl} \cdot T_{ik} - g_{il} \cdot T_{jk} - g_{jk} \cdot T_{il}\right)$$

where

$$T_{ij} := \nabla_{E_i} \nabla_{E_j} f - \nabla_{E_i} f \cdot \nabla_{E_j} f + \frac{1}{2} \cdot \|\underbrace{\operatorname{grad}}_{:= (df)^{\#} \in \Gamma(TM)} f\|_g^2 \cdot g_{ij}$$

$\#$ wrt g

In particular: if $g = (\delta_{ij})_{i,j}$, then

$$\tilde{R}_{ijji} = 0 - e^{2f} \cdot (T_{jj} - T_{ii}).$$

Proof. Computations... Ü 9.4. $\square$