26.11.2020

Recap: $S^n(R) = \{(x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_n^2 + t^2 = + R^2\} \subset \mathbb{R}^{n+1}$

homogeneous, isotropic, symmetric

1.4.4 HYPERBOLIC SPACES

idea: "dualise" the definition of $S^n(R)$

Proposition and Definition. (hyperbolic space). Let $n \in \mathbb{N}$, let $R \in \mathbb{R}_{>0}$. Then the following Riemannian manifolds are isometric, (notation $\mathbb{H}^n(R)$) called the n-dim hyperbolic space of radius R.

1. Hyperboloid model:
   - mfd: the smooth submfd

$$\mathbb{H}^n(R) := \{(x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_n^2 - t^2 = -R^2,\ t > 0\} \subset \mathbb{R}^{n+1}$$

   - Riemannian metric: In the coords $(x^1, \dots, x^n, t)$:

$$g^{\mathbb{H}} := \underbrace{(dx^1)^2 + \dots + (dx^n)^2 - (dt)^2}_{\substack{\text{not pos df. on } \mathbb{R}^{n+1} \\ \text{(but on } \mathbb{H}^n(R) \text{ - see below)}}}$$

2. Poincaré disk model:

   - mfd: $\mathbb{B}^n(R) := \{x \in \mathbb{R}^n \mid \|x\|_2 < R\} \subset \mathbb{R}^n$.
   - Riem. metric in the coords $x^1, \dots, x^n$:

$$g^{\mathbb{B}} := \frac{4 \cdot R^4}{(R^2 - \|x\|_2^2)^2} \cdot \big(\underbrace{(dx^1)^2 + \dots + (dx^n)^2}_{\text{Euclidean}}\big)$$

3. Poincaré upper halfspace model

- mfd: $U^n(R) := \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \subset \mathbb{R}^n$
- Riem metric in coords $x^1, \dots, x^{n-1}, y$: Euclidean

$$g^U := \frac{R^2}{y^2} \cdot \left((dx^1)^2 + \dots + (dx^{n-1})^2 + (\cancel{dx^n}\; dy)^2 \right)$$

Proof ① Smooth mfds? (of dim n)

- $B^n(R)$, $U^n(R)$: as open subsets of $\mathbb{R}^n$
- $\mathbb{H}^n(R)$: regular value thm, applied to

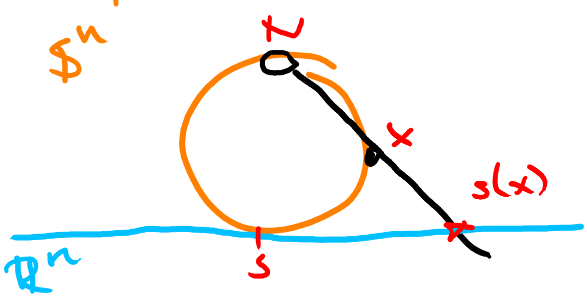
$$\mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}$$
$$(x,t) \longmapsto x_1^2 + \dots + x_n^2 - t^2 + R^2.$$

② Riemannian metrics?

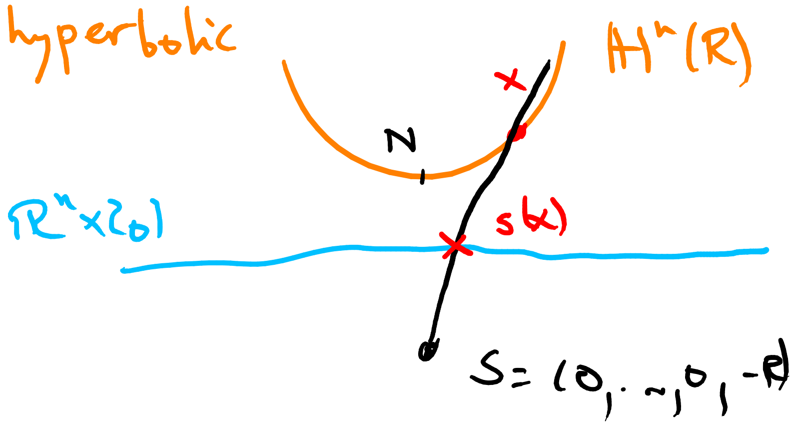
- g^B, g^U: rescalings of the Euclidean Riem. metric.
- g^H: we show that g^H is the pullback of g^B by a diffeo.

③ Comparison of hyperboloid model and the Poincaré disk model: We use hyperbolic stereographic projection:

spherical:



hyperbolic



Hyp. stereographic proj:

$$s: \mathbb{H}^n(R) \longrightarrow \mathbb{B}^n(R)$$
$$(x,t) \longmapsto \frac{R\cdot x}{\underbrace{R+t}_{>0}}.$$

- image of s lies in $\mathbb{B}^n(R)$: For all $(x,t) \in \mathbb{H}^n(R)$, we have

$$\boxed{\|s(x,t)\|_2^2} = \frac{R^2}{(R+t)^2}\cdot \overbrace{\|x\|_2^2}^{t^2-R^2} = \frac{R^2\cdot(t^2-R^2)}{(R+t)^2}$$
$$= \frac{R^2\cdot\overbrace{(t-R)}^{>0}}{(R+t)} \boxed{< R^2.}$$

- s is smooth ✓
- s has the following smooth inverse:

$$\mathbb{B}^n(R) \longrightarrow \mathbb{H}^n(R)$$
$$u \longmapsto \left(\frac{2\cdot R^2\cdot u}{R^2-\|u\|_2^2}, R\cdot\frac{R^2+\|u\|_2^2}{R^2-\|u\|_2^2}\right)$$

$\leadsto$ $s: \mathbb{H}^n(R) \longrightarrow \mathbb{B}^n(R)$ is a diffeo.

Moreover: $s^* g^{\mathbb{B}} = g^{\mathbb{H}}$:

Let $(x,t) \in \mathbb{H}^n(R)$. We show (suffices because of polarization)

$$\forall\, v \in T_{(x,t)}\mathbb{H}^n(R) \quad (s^* g^{\mathbb{B}})_{(x,t)}(v\otimes v) = g^{\mathbb{H}}_{(x,t)}(v\otimes v).$$

Let $v \in \underbrace{T_{(x,t)}\mathbb{H}^n(R)}$, say $v = (\overset{\in\mathbb{R}^n}{\xi}, \underset{\in\mathbb{R}}{\tau})$.

$s:(x,t)\mapsto \frac{R\cdot x}{R+t}$ "geometric" submfd tangent space

Then:

$$(s^* g^{\mathbb{B}})_{(x,t)}\big((\xi,\tau)\otimes(\xi,\tau)\big) = g^{\mathbb{B}}_{s(x,t)}\Big(\underbrace{d_{(x,t)}s(\xi,\tau)}_{=\frac{R}{R+t}\cdot\xi-\frac{R}{(R+t)^2}\cdot\tau\cdot x}\otimes d_{(x,t)}s(\xi,\tau)\Big)$$

$$g^{\mathbb{B}}_{s(x,t)} = \frac{4\cdot R^4}{(R^2-\|s(x,t)\|_2^2)^2}\cdot\text{Euclidean}$$

$$= \underbrace{\frac{4\cdot R^4}{(\underbrace{R^2-\|s(x,t)\|_2^2}_{=\frac{R^2\cdot(t-R)}{R+t}})^2}}_{=\cdots=1}\cdot\frac{R^2}{(R+t)^2}\cdot\underbrace{\Big\|\xi-\frac{1}{R+t}\cdot\tau\cdot x\Big\|_2^2}$$

$$= \langle\xi,\xi\rangle - 2\cdot\langle\xi,\tfrac{1}{R+t}\tau x\rangle + \frac{\tau^2}{(R+t)^2}\cdot\langle x,x\rangle$$

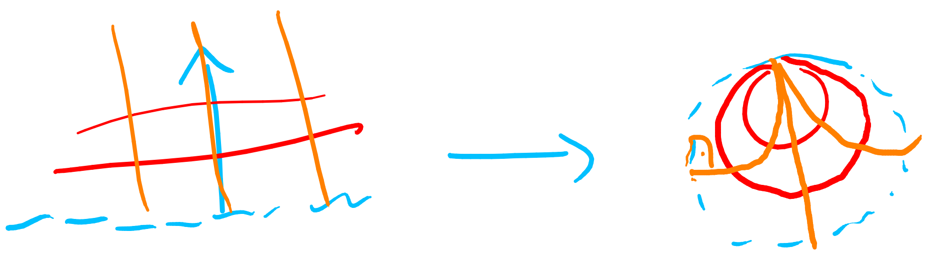
$$= \langle\xi,\xi\rangle + \frac{\tau^2}{(R+t)^2}\cdot\underbrace{\langle x,x\rangle}_{t^2-R^2} - \frac{2\tau}{R+t}\underbrace{\langle\xi,x\rangle}_{=\tau\cdot t}$$

because $(\xi,\tau)\in T_{(x,t)}\mathbb{H}^n(R)$

$$= \cdots = \langle\xi,\xi\rangle - \tau^2$$

$$= 1\cdot g^{\mathbb{H}}_{(x,t)}\big((\xi,\tau)\otimes(\xi,\tau)\big).$$

(4) Comparing the Poincaré halfspace model with the Poincaré disk model: Uses the Cayley transform



(Ü) 4.4

□

Proposition. Let $n \in \mathbb{N}$ and $R \in \mathbb{R}_{>0}$. Then hyperbolic n-space $\mathbb{H}^n(R)$ of radius R is homogeneous, isotropic, symmetric.

Proof. • homogeneity: We use the halfspace model: Two example classes of isometries of $(\mathbb{U}^n, g^{\mathbb{U}})$:

• "Horizontal" translation: For $a \in \mathbb{R}^{n-1}$, the map

$$\mathbb{U}^n(R) \longrightarrow \mathbb{U}^n(R)$$
$$(x,y) \longmapsto (x+a, y)$$

is a $g^{\mathbb{U}}$-isometry.

$$= \frac{R^2}{y^2} \cdot \left((dx^1)^2 + \dots + (dx^{n-1})^2 + (dy)^2\right)$$

• Scaling maps: For $\lambda \in \mathbb{R}_{>0}$, the map $f: \mathbb{U}^n(R) \longrightarrow \mathbb{U}^n(R)$, $z \longmapsto \lambda \cdot z$

is a $g^{\mathbb{U}}$-isometry: Let $(x, \overset{t}{\cancel{y}}) \in \mathbb{U}^n(R)$ and let $v \in T_{(x,\overset{t}{\cancel{y}})} \mathbb{U}^n(R)$. Then

$$\boxed{(f^* g^{\mathbb{U}})_{(x,t)}(v \otimes v)} = g^{\mathbb{U}}_{\underbrace{f(x,t)}_{=(\lambda x, \lambda t)}}\Big(\underbrace{d_{(x,t)} f(v)}_{=\lambda \cdot v} \otimes \underbrace{d_{(x,t)} f(v)}_{=\lambda \cdot v}\Big)$$

$$= \frac{R^2}{\lambda^2 \cdot t^2} \cdot \|\lambda \cdot v\|_2^2$$

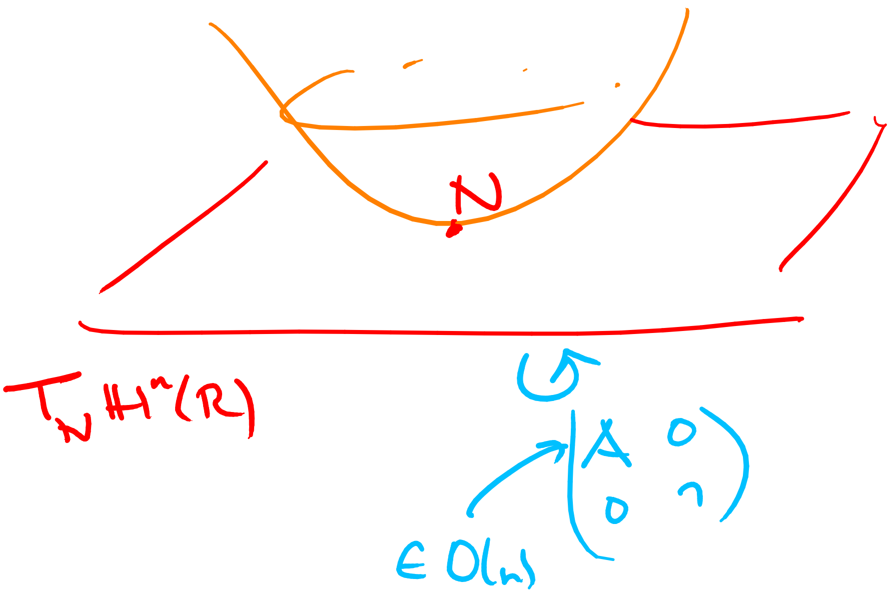
$$= \frac{R^2}{\lambda^2 t^2} \cdot \lambda^2 \cdot \|v\|_2^2 = \frac{R^2}{t^2} \cdot \|v\|_2^2$$

$$= \boxed{g^{\mathbb{U}}_{(x,t)}(v \otimes v)}.$$

- Transitivity: let $(x,y) \in \mathbb{U}^n(\mathbb{R})$, suffices to find an isometry that maps $(0,\dots,0,1)$ to (x,y).

  Can achieve this by:
  - scaling by y, and then
  - translating by x.

- isotropic / symmetric: We use the hyperboloid model. Because of homogeneity, we only need to this at a single point. We pick the north pole $N := (0,\dots,0,R)$.



- isotropic: use that $O(n)$ acts transitively on S^{n-1}
- symmetric: use $\begin{pmatrix} -E_n & 0 \\ 0 & 1 \end{pmatrix}$.

$\square$