27.01.2021

Recap: goal: understand the relation

geodesics $\longleftrightarrow$ (locally) minimising curves

$\boxed{3.2.3}$ MINIMISING CURVES ARE GEODESICS

Strategy: minimising curve

$\Downarrow$

critical point of L_g } $\leftarrow$ def ?!

$\Updownarrow$

geodesic

Definition (minimising curve). Let (M,g) be a Riem. mfd. A piecewise regular curve $\gamma : [a,b] \to M$ is <u>minimising</u> if

$\forall$ piecewise regular curve η with the same endpoints as γ: $\quad L_g(\eta) \geq L_g(\gamma)$.

Definition (critical point). Let (M,g) be a Riem. mfd. A piecewise regular unit speed curve γ on M is a <u>critical point of L_g</u> if for all proper variations G of γ, we have

$$\frac{\partial}{\partial s}\Big|_{s=0} L_g\left(G(s,\cdot)\right) = 0.$$

Theorem. Let (M,g) be a Riem. mfd, let γ be a pw regular unit speed curve on M. Then:

γ is a critical pt of L_g $\iff$ γ is a geodesic

Proof. Recall: first variation formula:

$$\frac{\partial}{\partial s}\Big|_{s=0} L_g(G(s,\cdot)) = -\int_a^b \langle \underset{\substack{\downarrow \\ \text{variation field of } G}}{V}, D_\gamma \dot\gamma\rangle_g - \sum_{j=1} \langle V(a_j), \underset{\substack{\downarrow \\ \text{jump}}}{\Delta_j}\dot\gamma\rangle_g$$

"$\Longleftarrow$" Let γ be a geodesic. Let G be a proper variation of γ. Then

$$\frac{\partial}{\partial s}\Big|_{s=0} L_g(G(s,\cdot)) = -\int_a^b \langle V, \underbrace{D_\gamma \dot\gamma}_{=0 \text{ (γ geodesic!)}}\rangle_g - \overbrace{\sum_{j=1}^{k-1} \langle V(a_j), \Delta_j \dot\gamma\rangle_g}^{=0 \text{ (γ smooth!)}}$$

$$= 0.$$

"$\Longrightarrow$" Let γ be a critical point of L_g.

Goal: show that γ is a geodesic. Strategy:

① Show that γ has the geodesic property on the smooth part.

② Show that γ is smooth.

Use FVF!

Proof of ①: Let $[\tilde a, \tilde b] \subset [a,b]$ on which γ is smooth. Use a bump function to localise the problem!

Let $\varphi : [a,b] \to \mathbb{R}$ be a smooth function with

$$\varphi|_{(\tilde a, \tilde b)} > 0 \quad \text{and} \quad \varphi|_{[a,b]\setminus(\tilde a, \tilde b)} = 0.$$

Consider: $V := \varphi \cdot D_\gamma \dot\gamma$.

Then: there ex. a proper variation G of γ with variation field is V.

We obtain:

γ critical pt of L_g

$$0 = \frac{\partial}{\partial s}\Big|_{s=0} L_g(G(\cdot,\cdot))$$

FVF $\rightsquigarrow$

$$= -\int_a^b \langle V, \underbrace{D_\gamma \dot\gamma}_{\varphi \cdot \langle D_\gamma\dot\gamma, D_\gamma\dot\gamma\rangle_g} \rangle_g - \underbrace{\sum_{j=1}^{k-1} \langle \underbrace{V(a_j)}_{=0}, \Delta_j \dot\gamma \rangle_g}_{=0}$$

$$= -\int_{\tilde a}^{\tilde b} \underbrace{\varphi}_{>0 \text{ on } (\tilde a, \tilde b)} \cdot \underbrace{\|D_\gamma \dot\gamma\|_g^2}_{\geq 0}$$

Thus $\underbrace{\|D_\gamma \dot\gamma(t)\|_g^2 = 0}$ for all $t \in (\tilde a, \tilde b)$.

$\rightsquigarrow D_\gamma \dot\gamma(t) = 0$

Proof of ②: similar, use ① and vector fields that focus on single singular pts. $\square$

Corollary. (minimising curves are geodesics). Let (M, g) be a Riem mfd. Then: every unit speed minimising curve on M is a Riem. geodesic.

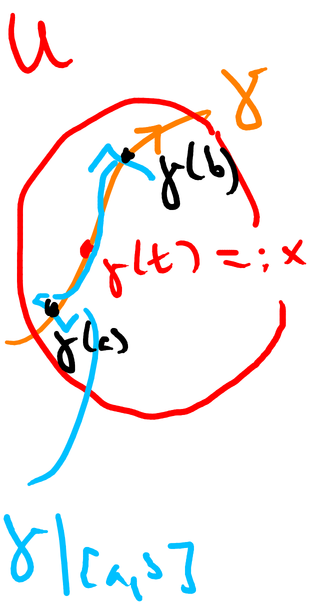
Proof. Let $\gamma: [a,b] \to M$ be a minimising unit speed curve. Then γ is a critical pt: If G is a variation of γ, then $s \mapsto L_g(G(s,\cdot))$ is smooth and has a minimum at 0. Thus 0 is a critical pt of this function. Now apply the thm. $\square$

B.2.4 Geodesics are locally minimising

idea: work in uniformly normal nbhds / normal coord's, solve it for radial geodesics.

Theorem. Let (M,g) be a Riem mfd and let γ be a geodesic on M. Then: γ is locally minimising and γ has constant speed.

Proof. Let $\gamma : I \to M$, where I is open, let $t \in I$.



Let $U \subset M$ be a uniformly normal nbhd around x, let J be the connected component of $\gamma^{-1}(U)$ with $t \in J$. Let $a, b \in J$ with $a < t < b$.

As U is uniformly normal, $\gamma(b)$ lies in a normal nbhd of $\gamma(a)$.

Then: the radial geodesic from $\gamma(a)$ to $\gamma(b)$ is simultaneously

→ ① (up to affine repar.) the only geodesic from $\gamma(a)$ to $\gamma(b)$ inside U

→ ② and a minimising curve from $\gamma(a)$ to $\gamma(b)$.

By ①: γ has to be a radial geodesic from $\gamma(a)$ to $\gamma(b)$.

By ②: this is minimising. $\square$

Proof of ⓪: geodesics γ have constant speed:

$$\forall_t \quad \left(\|\dot\gamma\|_g^2\right)'(t) = \langle \dot\gamma, \dot\gamma\rangle_g'(t)$$

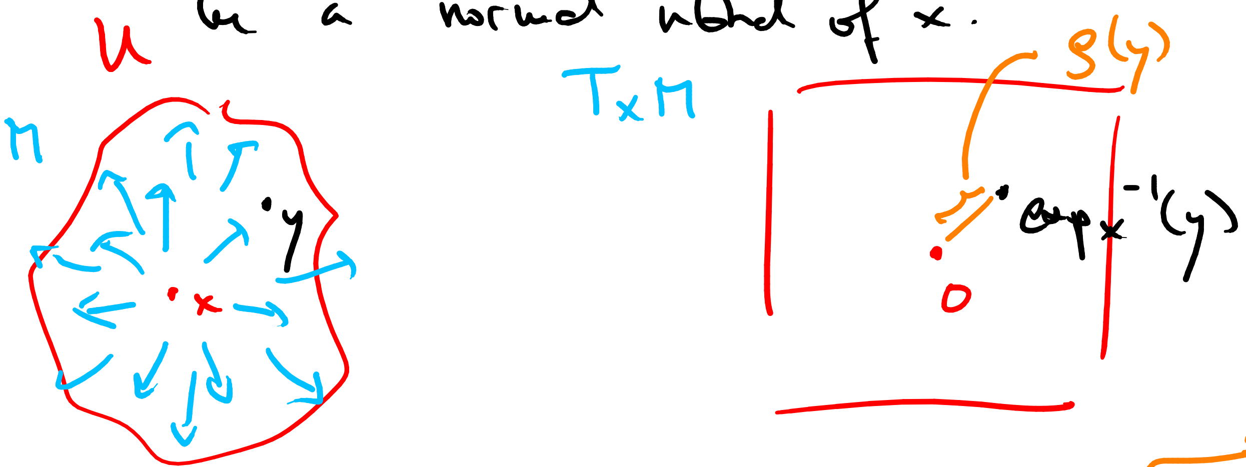
$$= 2 \cdot \langle \underbrace{D_\gamma \dot\gamma(t)}_{=0}, \dot\gamma(t)\rangle_g = 0.$$

$\rightsquigarrow$ $\|\dot\gamma\|_g$ is constant.

Proof of ①: use properties of normal nbhds and $\exp_{\gamma(a)}$ (Prop. 3.2.15).

Proof of ②: We need some preparations:

Definition. (unit radial vector field). Let (M,g) be a Riem. mfd, let $x \in M$, and let $U \subset M$ be a normal nbhd of x.



- The radial distance function at x is def'd as
$$\rho : U \longrightarrow \mathbb{R}_{\geq 0}$$
$$y \longmapsto \|\exp_x^{-1}(y)\|_g.$$

- The <u>unit radial vector field at x</u> is

(!)
$$\frac{\partial}{\partial \varrho} : U \setminus \{x\} \longrightarrow TM$$
$$y \longmapsto \frac{1}{\varrho(y)} \cdot \dot{\gamma}(1),$$
where $\gamma := \mathrm{geod}_{x, \exp_x^{-1}(y)}$

Proposition. (Gauß lemma). Let (M,g) be a Riem mfd, let $x \in M$, let $U \subset M$ be a normal nbhd, centered at x. Then: all geodesic spheres around x that lie in U are orthogonal to $\frac{\partial}{\partial \varrho}$ (at x).

Proof. Let $y \in U \setminus \{x\}$, let $v \in T_y M$ be <u>tangent</u> to <u>sphere</u> (x). Goal: $\langle v, \frac{\partial}{\partial \varrho}(y) \rangle_g = 0$.

$\underbrace{\varrho(y)}_{=: R}$

Let $w := \exp_x^{-1}(y)$, let $\tilde{v} \in T_x M \cong T_w(T_x M)$ be s.t. $d_w \exp_x(\tilde{v}) = v$.

Let $\gamma := \mathrm{geod}_{x,w}$. $\leadsto$ $\dot{\gamma}(1) = R \cdot \frac{\partial}{\partial \varrho}(y)$.

$\leadsto$ suffices to show $\langle v, \dot{\gamma}(1) \rangle_g = 0$.

Let $\eta : (-\varepsilon, \varepsilon) \to T_x M$ be a smooth curve in $\partial B_R^{\|\cdot\|_g}(0) \subset T_x M$ with
$$\eta(0) = w \quad \text{and} \quad \dot{\eta}(0) = \tilde{v}.$$

We consider the smooth variation: (of γ)

$$G : (-\varepsilon, \varepsilon) \times I \longrightarrow M \quad \text{(domain of } \gamma \text{ in } \mathrm{Exp}_x)$$

$$(s,t) \longmapsto \exp_x\big(t \cdot \eta(s)\big) = \mathrm{geod}_{x,\eta(s)}(t).$$

and $S := \partial_{①} G$, $T := \partial_{②} G$. Then

$$\langle S(0,0), T(0,0)\rangle_g \overset{(\dots)}{=} 0$$

$$\langle S(0,1), T(0,1)\rangle_g \overset{(\dots)}{=} \langle v, \dot\gamma(1)\rangle_g .$$

$\rightsquigarrow$ suffices to show that $\langle S(0,\cdot), T(0,\cdot)\rangle_g$ is constant,

Thus: show that the derivative is 0.

(compat. of ∇ with g, symmetry lemma for variations, γ has unit speed). $\square$

Corollary. Let (M,g) be a Riem. mfd, let $x \in M$, let $U \subset M$ be a normal nbhd of x. Then: on $U \setminus \{x\}$, we have

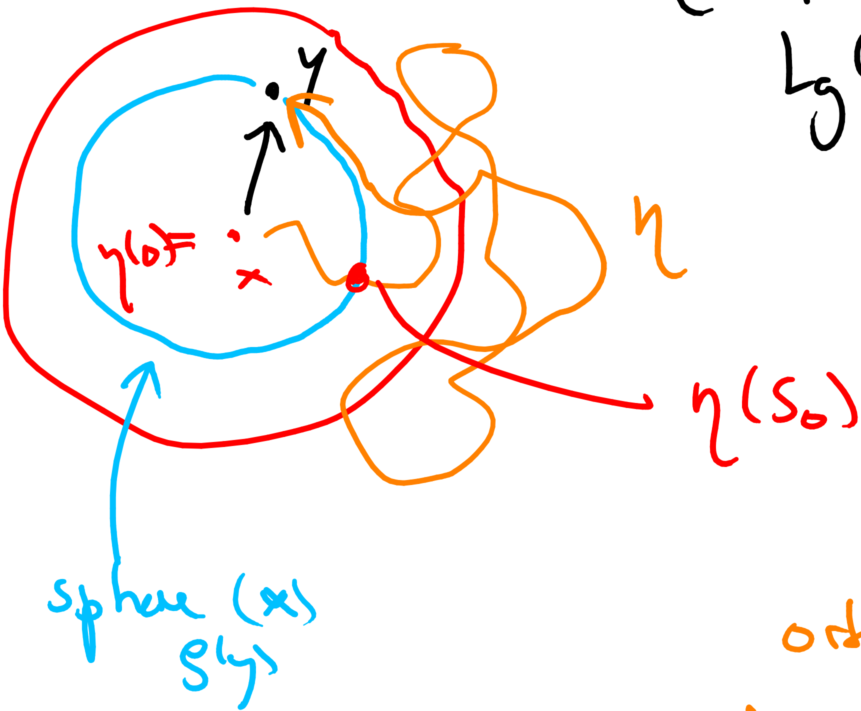
$$\operatorname{grad} \varrho = \frac{\partial}{\partial \varrho}.$$

Proof. Key ingredient: By the Gauß lemma, $\frac{\partial}{\partial \varrho}$ is orthogonal to the $\underbrace{\text{geodesic spheres}}_{\text{level sets of } \varrho}$ $\square$

Corollary. Let (M,g) be a Riem. mfd, let $x \in M$, let $U \subset M$ be a normal nbhd open geod. ball around x, and let $y \in U \setminus \{x\}$.

Then: the radial geodesic from x to y is minimizing (unique up to affine repar).

Idea of proof. For every pw regular curve η from x to y:



$$L_g(\eta) \geq L_g(\underbrace{\eta|_{[0,s_0]}}_{\text{lives in } U})$$

Gauss lemma: orth. decomposition into

- radial part $\left(\frac{\partial}{\partial \rho}\right)$
- part tangential to the geodesic spheres

$(\ldots)$

$$\geq L_g(\text{radial geod. from } x \text{ to } y). \quad \square$$