28.01.2021

Recap: minimising $\Rightarrow$ geodesic

geodesic $\Rightarrow$ locally minimising

radial geodesics in normal nbhd balls are min.

Now: understand more about

Riem. geometry $\leftrightarrow$ metric geometry.

3.2.5 RIEMANNIAN ISOMETRIES

Theorem. Let (M,g) be a connected Riem. mfd. Then the Riem isometry group $\mathrm{Isom}(M,g)$ coincides with the metric isometry group $\mathrm{Isom}(M, d_g)$.

Proof: • $\mathrm{Isom}(M,g) \subset \mathrm{Isom}(M,d_g)$

Prop. 1.5.8

• $\supset$: use geodesics! (...) + exp $\square$

Corollary. The sphere S^2 with the spherical metric is not locally metrically isometric to $\mathbb{R}^2$ with the Euclidean metric. $\square$

Thm + curvature

$\rightsquigarrow$ cartography problem!

3.3 Completeness

(M,g) Riem mfd $\rightsquigarrow$ metric completeness $\Leftrightarrow$ geodesic completeness (Hopf-Rinow thm)

Definition. (completeness). Let (M,g) be a Riem mfd.

- (M,g) is <u>metrically complete</u> if M is complete as a metric space wrt d_g.
- (M,g) is <u>geodesically complete</u> if every maximal geodesic on (M,g) is defined on all of $\mathbb{R}$.

Example:

- $\mathbb{R}^n$ with the Euclid. Riem. metric is metrically complete and geodesically complete.
- $\mathbb{R}^2\setminus\{0\}$ with the Euclid. Riem. metric is <u>not</u> metrically complete and <u>not</u> geodesically complete.

$\mathbb{R}^2\setminus\{0\}$

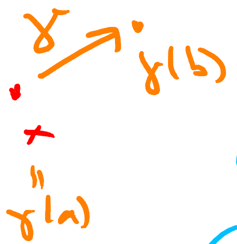
$(-1,0)$

Theorem. (Hopf-Rinow). Let (M,g) be a connected Riem mfd with $M\neq\emptyset$. Then the following are equivalent:

1. (M,g) is geodesically complete
2. For all $x\in M$: $\mathrm{Exp}_x = T_xM$.
3. There ex. $x\in M$ with $\mathrm{Exp}_x = T_xM$.
4. (M,g) is metrically complete

Key Proposition. Let (M,g) be a connected Riem mfd and let $x \in M$ with $\mathrm{Exp}_x = T_xM$.
Then: for any $y \in M$, there ex. a minimizing geodesic from x to y.

Proof. Let $y \in M$. A geodesic $\gamma : [a,b] \to M$ <u>aims at y</u> if γ is minimizing and if $\gamma(a) = x$ and

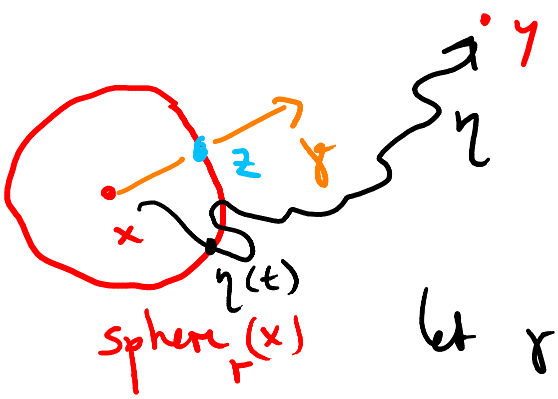


$$d_g(x,y) = d_g(x, \gamma(b)) + d_g(\gamma(b), y).$$

Goal: show that there ex. a geodesic
(*) starting at x that aims at y and has length $d_g(x,y)$. $\Big\} \Rightarrow$ ends at y.
(this suffices!)

Strategy: start with a short radial geodesic in the "right" direction and then extend.

Let $r \in \mathbb{R}_{>0}$ be so small that $\overline{\mathrm{ball}_r(x)}$ lies in a normal nbhd of x. (wlog $y \notin \overline{\mathrm{ball}_r(x)}$)



Let $z \in \mathrm{sphere}_r(x)$ be a pt that in $\mathrm{sphere}_r(x)$ minimizes $d_g(\cdot, y)$. (ex!)

Let $\gamma : I \to M$ be the max. unit speed radial geodesic from x through z.

Claim: ① $\gamma|_{[0,r]}$ aims at y

② $\gamma|_{[0,\, d_g(x,y)]}$ aims at y. $\rightsquigarrow$ (*)

Proof of ①: • $\gamma|_{[0,r]}$ is minimizing (Cor. 3.2.21)

• $d_g(x,y) = d_g(x,z) + d_g(z,y)$:

$\leq$: Δ-ineq.

$\geq$: let $\eta : [a,b] \to M$ be a pw regular curve from x to y.

Let $t \in [a,b]$ the first time that η hits $\text{sphere}_r(x)$. Then

$$\boxed{L_g(\eta)} = L_g(\eta|_{[a,t]}) + L_g(\eta|_{[t,b]})$$

def of d_g $\geq$ $\underbrace{d_g(x,\eta(t))}_{=r} + \underbrace{d_g(\eta(t),y)}_{\geq d_g(z,y)}$

$$\geq \boxed{d_g(x,z) + d_g(z,y)}.$$

take $\inf_\eta$.

Proof of ②: standard extension argument using uniformity normal nblds and ①. □

Proof of the Hopf-Rinow thm:

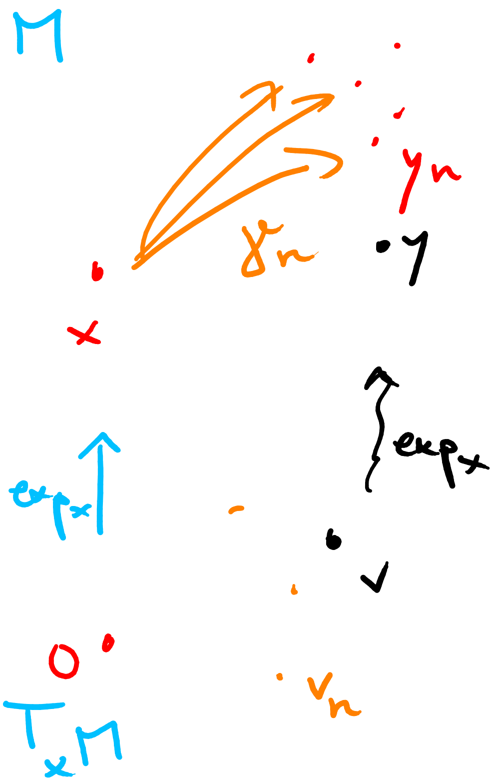
1. geod. complete 2. $\forall_x \ \mathrm{Exp}_x = T_xM$ 3. $\exists_x \ \mathrm{Exp}_x = T_xM$

4. metr. complete

1. $\Longrightarrow$ 2. by def.

2. $\Longrightarrow$ 3. because $M \neq \emptyset$.

3. $\Longrightarrow$ 4. let $x \in M$ with $\mathrm{Exp}_x = T_xM$. let $(y_n)_{n\in\mathbb{N}}$ be a d_g-Cauchy seq in M.



By the key prop., for each $n \in \mathbb{N}$, there ex. a unit speed min. geodesic

$$\gamma_n : [0, \underbrace{D_n}_{d_g(x,y_n)}] \longrightarrow M \text{ from } x \text{ to } y_n.$$

There ex. $v_n \in T_xM$ with

$$\gamma_n = \mathrm{geod}_{x,v_n}\big|_{[0,D_n]} \text{ and } \|v_n\|_g = 1.$$

Then

$$(D_n \cdot v_n)_{n\in\mathbb{N}} \subset T_xM$$

is bounded.

$\rightsquigarrow$ ex. accumulation point $v \in T_xM$.

let $y := \exp_x(v)$. $\in M$

$\exp_x$ cont.

$\rightsquigarrow$ y is an accumulation pt of $\big(\overbrace{\exp_x(D_n \cdot v_n)}^{= y_n}\big)_{n\in\mathbb{N}}$

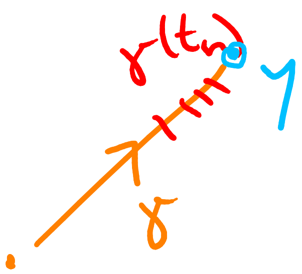
$(y_n)_{n\in\mathbb{N}}$ Cauchy

$\rightsquigarrow$ y is a d_g-limit of $(y_n)_{n\in\mathbb{N}}$.

4. $\Rightarrow$ 1. Let (M,g) be metrically complete.

Assume for a contradiction that (M,g) is <u>not</u> geodesically complete. Thus: there ex. a unit speed geodesic $\gamma : I \to M$ that does not extend beyond $b := \sup I < \infty$.

Let $(t_n)_{n\in\mathbb{N}} \subset I^\circ$ be an increasing seq with $\lim_{n\to\infty} t_n = b$.



· $(\overbrace{y_n}^{\gamma(t_n)})_{n\in\mathbb{N}}$ is a d_g-Cauchy seq:

$$\forall_{\substack{n,m\in\mathbb{N}\\ n<m}} \quad d_g(y_n, y_m) \le L_g(\gamma|_{[t_n,t_m]}) \overset{\gamma \text{ unit speed}}{=} \underbrace{|t_n - t_m|}_{\xrightarrow[n,m\to\infty]{} 0}$$

M is d_g-complete $\rightsquigarrow$ this seq. $(y_n)_{n\in\mathbb{N}}$ d_g-converges to a point $y \in M$.

Now: standard extension argument in a uniformly normal nbhd of y by radial geodesics. $\square$

Corollary. Let (M,g) be a connected Riem mfd.

Then: Hopf-Rinow $\Rightarrow$ (M,g) geod. complete $\Downarrow$ Key Prop.

If $\underbrace{(M,g) \text{ is metrically complete}}_{\Uparrow \; M \text{ compact}}$, then each two points in M can be connected by a min geodesic. $\square$

3.4 Model Spaces

goal: determine the images of geodesics in the model spaces.

Proposition. (fixed sets are geodesics). Let (M,g) be a Riem mfd and let $N \subset M$ be a connected one-dim smooth submfd for which there ex. $\varphi \in \mathrm{Isom}(M,g)$ with

$$N = \{x \in M \mid \varphi(x) = x\}.$$

Let $x \in N$. Then: for each of the two unit vectors $v \in T_xN \subset T_xM$, we have

$$\mathrm{im}(\mathrm{geod}_{x,v}) = N.$$

Proof. (Li) 10.4. $\square$

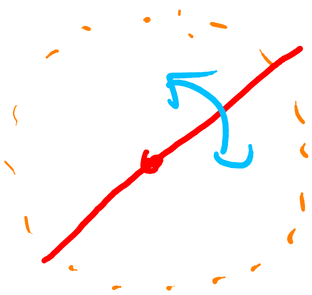
Example. In $\mathbb{R}^n$ (with the Euclid. Riem metric), the images of max. geodesics are exactly the affine lines.

Example. Let $n \in \mathbb{N}_{>0}$. Then the images of max geodesics of $\mathbb{S}^n$ (with the round Riem metric) are exactly the great circles of $\mathbb{S}^n$:

- Each great circle is the image of a max. geod.: by the prop. (take the orth. reflection at the plane of the great circle).
- Why can't there be more? The great circles already sweep out all possible initial conditions for geodesics.

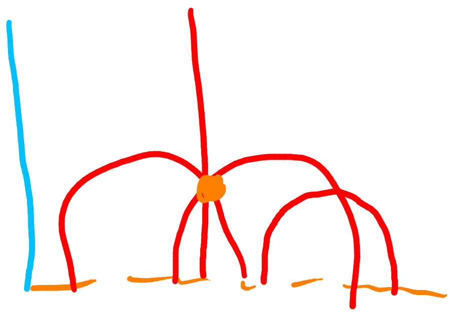
Example. hyperbolic n-space

- Images of max. geodesics passing through the centre of the Poincaré disk model are exactly the lines through 0. (argument as for S^n)



$\Big\updownarrow$ Cayley transform

- Images of max. geod. in the half-space model are exactly the vertical lines and the semi-circles with centre on the "lower plane".



$\rightsquigarrow$ $\mathbb{H}^2$ does not satisfy the parallel postulate.