Differential Geometry I: Week 2

Prof. Dr. C. Löh/AG Ammann November 5, 2020

Reading assignment (for the lecture on November 11). We finish our quick recap of smooth manifolds by recalling submanifolds. After that, we will introduce the language of (smooth) vector bundles.

- Read Chapter 1.1.5 *Submanifolds*.
- Read Chapter 1.2.1 *Smooth vector bundles*.
- Read Chapter 1.2.2 *Constructing vector bundles* until Proposition 1.2.7.

Reading assignment (for the lecture on November 12).

- Read the rest of Chapter 1.2.2 *Constructing vector bundles*.
- Read Chapter 1.2.3 *The tangent bundle*.
- Read Appendix A.1 *Categories and functors*. We will not use categories and functors in a serious way, but it is convenient to be able to use this language now and then.

Next week, we will construct tensor bundles of smooth manifolds and finally we will introduce Riemannian metrics.

Étude (regular values). Which of the following values are regular values of the given maps?

1. 2020 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $x \longmapsto \|x\|_2{}^2$
2. 2020 of $M_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}$, $A \longmapsto \operatorname{tr} A$
3. 2020 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto y^2 - x^3$
4. 0 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto y^2 - x^3$
5. 2020 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto x \cdot y$
6. 0 of $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto x \cdot y$

Exercises (for the session on November 16/17). The following exercises (which all are solvable with the material read/discussed in week 1) will be discussed.

Please turn over

Exercise 1.1 (smooth charts). Let M be a smooth manifold of dimension n and let $U \longrightarrow U'$ and $V \longrightarrow V'$ be smooth charts of M. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. We have $U' \cap V' \cong_{\mathsf{Top}} U \cap V$.
2. If $U \cong_{\mathsf{Top}} \mathbb{R}^n$ and $V \cong_{\mathsf{Top}} \mathbb{R}^n$, then $U \cap V = \emptyset$ or $U \cap V \cong_{\mathsf{Top}} \mathbb{R}^n$.

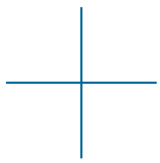
Exercise 1.2 (local vs. global diffeomorphisms). A smooth map $f\colon M \longrightarrow N$ between smooth manifolds is a *local diffeomorphism* if for each $x \in M$, there exists an open neighbourhood $U \subset M$ of x in M such that $f(U) \subset N$ is open and $f|_U\colon U \longrightarrow f(U)$ is a diffeomorphism.

1. Show that not every surjective local diffeomorphism is a diffeomorphism.
2. Show that bijective local diffeomorphisms are diffeomorphisms.

Exercise 1.3 (a non-manifold). Show that the set

$$\bigl\{(x,y) \in \mathbb{R}^2 \bigm| x \cdot y = 0\bigr\} \subset \mathbb{R}^2$$

is *not* a topological manifold with respect to the subspace topology inherited from the standard topology on $\mathbb{R}^2$. Illustrate your arguments with suitable pictures!



Exercise 1.4 (smooth currying). Let V and W be finite-dimensional $\mathbb{R}$-vector spaces, let M be a smooth manifold, and let $f\colon M \times V \longrightarrow W$ be a smooth map with the property that for each $x \in M$, the map $f(x, \cdot\,)\colon V \longrightarrow W$ is $\mathbb{R}$-linear. Show that then also the "curried" function

$$\begin{aligned} M &\longrightarrow \operatorname{Hom}_{\mathbb{R}}(V,W) \\ x &\longmapsto f(x, \cdot\,) \end{aligned}$$

is smooth.
Hints. It might be helpful to think about this in terms of bases.

Bonus problem (the category of smooth manifolds). Justify your answers with suitable arguments!

1. Does the category Mfd of smooth manifolds have an initial object?
2. Does the category Mfd of smooth manifolds have a terminal object?
3. Does the category Mfd of smooth manifolds contain all finite products?
4. Does the categoyr Mfd of smooth manifolds contain all finite coproducts?

Submission before November 12, 2020, 10:00, via email to your tutor.