Differential Geometry I: Week 3

Prof. Dr. C. Löh/AG Ammann November 12, 2020

Reading assignment (for the lecture on November 18). Applying constructions from multilinear algebra to the tangent bundle leads to the tensor bundles of the tangent bundle, which allow to define Riemannian metrics.

- Recall dual spaces, tensor products, and exterior products, including universal properties, constructions, and how to find bases.
- Read Chapter 1.2.4 *Tensor bundles.*
- Read Chapter 1.3.1 *Riemannian metrics* until Quick check 1.3.2.

Reading assignment (for the lecture on November 19).

- Read Appendix A.2 *Partitions of unity* to recall/learn this tool.
- Read the rest of Chapter 1.3.1 *Riemannian metrics.*
- Read Chapter 1.3.2 *Riemannian manifolds.*

Next week, we will spend some time with the three model spaces of Riemannian geometry and start with the study of Riemannian *geometry.*

Étude (Riemannian metrics). Which of the following terms define Riemannian metrics on $\mathbb{R}^2$?

1. $\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^2$
2. $\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1$
3. $\sin(x^1 + x^2) \cdot \mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1 + \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^2$
4. $\frac{1}{e^{x^1} + x^1 \cdot x^1} \cdot (\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1 + \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^2)$
5. $\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1 - \mathrm{d}\,x^1 \otimes \mathrm{d}\,x^2 - \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^1 + 2 \cdot \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^2$
6. $\mathrm{d}\,x^1 \otimes \mathrm{d}\,x^1 - \mathrm{d}\,x^1 \otimes \mathrm{d}\,x^2 - \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^1 + \mathrm{d}\,x^2 \otimes \mathrm{d}\,x^2$

Exercises (for the session on November 23/24). The following exercises (which all are solvable with the material read/discussed in week 2) will be discussed.

Please turn over

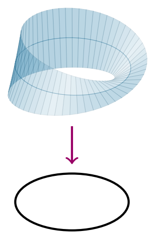
Exercise 2.1 (isomorphic vector bundles). Let M be a smooth manifold and let $\pi: E \longrightarrow M$ and $\pi': E' \longrightarrow M$ be smooth vector bundles over M. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If π and π' are isomorphic smooth vector bundles, then $E_x \cong_{\mathbb{R}} E'_x$ for all $x \in M$.
2. If $E_x \cong_{\mathbb{R}} E'_x$ holds for all $x \in M$, then π and π' are isomorphic smooth vector bundles.

Exercise 2.2 (trivial vector bundles). Let $\pi: E \longrightarrow M$ be a smooth vector bundle of rank $k \in \mathbb{N}$. Show that the following are equivalent:

- The vector bundle π is trivial.
- The vector bundle admits k linearly independent sections, i.e., there exist sections $s_1, \dots, s_k \in \Gamma(\pi)$ with the property that for all $x \in M$, the family $(s_1(x), \dots, s_k(x))$ in E_x is linearly independent over $\mathbb{R}$.

Exercise 2.3 (Möbius strip). We can view the open Möbius strip as a line bundle over the circle $\mathbb{S}^1$ (see below). Provide a rigorous construction of this line bundle via a suitable cocycle (two patches suffice). Illustrate your arguments with suitable pictures!
Hints. It is instructive to build a paper Möbius strip and to work out bundle theory on this model.

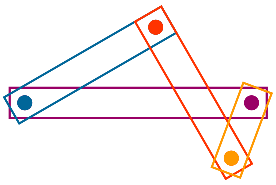


Exercise 2.4 (Möbius strip, embedded). Specify a smooth submanifold of $\mathbb{R}^3$ concretely that is diffeomorphic to the Möbius strip (and prove this fact).
Hints. You can easily visually check whether your subset of $\mathbb{R}^3$ is a valid candidate by plotting it with a suitable program!

Bonus problem (mechanical linkages). We consider the following article:

> M. Kapovich, J. Millson. Universality theorems for configuration spaces of planar linkages, *Topology*, 41(6), pp. 1051–1107, 2002.

1. How can compact smooth manifolds be represented in terms of mechanical linkages? It suffices to briefly introduce the main notions and to cite appropriate results from the article!
2. Does every mechanical linkage lead to a compact smooth manifold?



Submission before November 19, 2020, 10:00, via email to your tutor.