Differential Geometry I: Week 4

Prof. Dr. C. Löh/AG Ammann November 19, 2020

Reading assignment (for the lecture on November 25). We will introduce important families of "symmetric" examples of Riemannian manifolds: spheres, Euclidean spaces, hyperbolic spaces.

- Recall the notion of *group actions* (Appendix A.3).
- Read Chapter *1.4.1 Homogeneous spaces*.
- Read Chapter *1.4.2 Euclidean spaces*.
- Read Chapter *1.4.3 Spheres*.

Reading assignment (for the lecture on November 26).

- Read Chapter *1.4.4 Hyperbolic spaces*.
- (Optional) Get used to the hyperbolic plane by playing *HyperRogue*: https://github.com/zenorogue/hyperrogue

It is time to turn Riemannian geometry into actual geometry: Next week, we will outline the construction of geometric invariants via Riemannian metrics.

Étude (Riemannian isometries). Let g° denote the round Riemannian metric on $\mathbb{S}^1$ and let g be the Euclidean Riemannian metric on $\mathbb{R}^2$. Which of the following maps are local isometries? Isometries?

1. The inclusion $(\mathbb{S}^1, g^\circ) \longrightarrow (\mathbb{R}^2, g)$.
2. The doubling map
$$\begin{aligned}(\mathbb{S}^1, g^\circ) &\longrightarrow (\mathbb{S}^1, g^\circ)\\ (\cos t, \sin t) &\longmapsto \bigl(\cos(2\cdot t), \sin(2\cdot t)\bigr).\end{aligned}$$
3. The doubling map
$$\begin{aligned}(\mathbb{S}^1, 2\cdot g^\circ) &\longrightarrow (\mathbb{S}^1, g^\circ)\\ (\cos t, \sin t) &\longmapsto \bigl(\cos(2\cdot t), \sin(2\cdot t)\bigr).\end{aligned}$$
4. The map
$$\begin{aligned}(\mathbb{R}^2, g) &\longrightarrow (\mathbb{R}^2, g)\\ (x, y) &\longmapsto (x^2, y^2).\end{aligned}$$

Exercises (for the session on November 30/December 1). The following exercises (which all are solvable with the material read/discussed in week 3) will be discussed.

Please turn over

Exercise 3.1 (Riemannian metrics on $\mathbb{R}^2$). Let g be a Riemannian metric on $\mathbb{R}^2$ and let x^1, x^2 be the standard coordinates on $\mathbb{R}^2$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. There exist $f_{11}, f_{22} \in C^\infty(\mathbb{R}^2)$ with
$$g = f_{11} \cdot (\mathrm{d}\, x^1)^2 + f_{22} \cdot (\mathrm{d}\, x^2)^2.$$
2. There exist $f_{11}, f_{21}, f_{22} \in C^\infty(\mathbb{R}^2)$ with
$$g = f_{11} \cdot (\mathrm{d}\, x^1)^2 - f_{21} \cdot \mathrm{d}\, x^2 \cdot \mathrm{d}\, x^1 + f_{22} \cdot (\mathrm{d}\, x^2)^2.$$

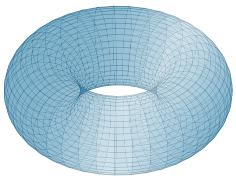
Exercise 3.2 (scaling Riemannian metrics). Let M be a smooth manifold, let $g_1, g_2 \in \mathrm{Riem}(M)$, and let $f_1, f_2 \in C^\infty(M, \mathbb{R}_{\geq 0})$ with the property that $f_1 + f_2 > 0$ (pointwise). Moreover, let

$$\begin{aligned} g := f_1 \cdot g_1 + f_2 \cdot g_2 \colon M &\longrightarrow \mathbf{T}^{2,0} M \\ x &\longmapsto \big(v \otimes w \mapsto f_1(x) \cdot (g_1)_x(v \otimes w) + f_2(x) \cdot (g_2)_x(v \otimes w)\big) \end{aligned}$$

Show that g is a Riemannian metric on M.

Exercise 3.3 (two 2-tori). Let g° denote the round Riemannian metric on $\mathbb{S}^1$.

1. Show that the Riemannian manifold $(\mathbb{S}^1 \times \mathbb{S}^1, g^\circ \oplus g^\circ)$ is locally isometric to $\mathbb{R}^2$ with the Euclidean Riemannian metric.
2. Find a smooth submanifold $T \subset \mathbb{R}^3$ that is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.
3. *Bonus problem.* Is your diffeomorphism isometric with respect to $g^\circ \oplus g^\circ$ and the first fundamental form of T in $\mathbb{R}^3$?



Exercise 3.4 (existence of local orthonormal frames). Let (M, g) be a Riemannian manifold and let $x \in M$. Show that there exists an open neighbourhood $U \subset M$ of x that admits an orthonormal frame.

An *orthonormal frame on U* is a family $(s_1, \dots, s_n)$ of sections $U \longrightarrow \mathrm{T}\, M$ of the tangent bundle of M over U (where $n := \dim M$) with the following property: For each $y \in U$, the family $(s_1(y), \dots, s_n(y))$ in $\mathrm{T}_y\, M$ is an orthonormal basis for $\mathrm{T}_y\, M$ with respect to g_y.

Hints. Gram-Schmidt might help! You should *not* aim at proving that there always exist local orthonormal *coordinate* frames (because this is wrong ...).

Bonus problem (visualisation of Riemannian metrics). Devise a program that visualises Riemannian metrics on $\mathbb{R}^2$:

1. How could one visualise Riemannian metrics on $\mathbb{R}^2$?

   Hints. What about drawing the shape/size of unit circles in the tangent planes at different points of $\mathbb{R}^2$?

2. Implement as much of this as you can.

Submission before November 26, 2020, 10:00, via email to your tutor.