Differential Geometry I: Week 5

Prof. Dr. C. Löh/AG Ammann November 26, 2020

Reading assignment (for the lecture on December 2)**.** We will first see that good actions by symmetries lead to interesting examples of Riemannian manifolds via quotients. Then, we will switch to extracting geometric data and invariants from Riemannian metrics, starting with the length of curves.

- Read Chapter *1.4.5 Group actions*.
- Read Chapter *1.5.1 Lengths of curves*.

Reading assignment (for the lecture on December 3)**.** Using length of curves, we can introduce the Riemannian distance function on Riemannian manifolds.

- Read Chapter *1.5.2 The Riemannian distance function*.
- As a preparation for next week: Recall differential forms on smooth manifolds and their integration.

Next week, we will quickly review orientations and volume and then start developing the analytic foundations for curvature of Riemannian manifolds.

Étude (lengths of curves)**.** Compute the lengths of the following curves:

1. $[0, 2\pi] \longrightarrow \mathbb{S}^2,\ t \longmapsto (\cos t, \sin t, 0)$ in $\mathbb{S}^2$ with the round metric.
2. $[0, 4\pi] \longrightarrow \mathbb{S}^2,\ t \longmapsto (\cos t, \sin t, 0)$ in $\mathbb{S}^2$ with the round metric.
3. $[0, 2\pi] \longrightarrow \mathbb{S}^2(2),\ t \longmapsto 2 \cdot (0, \cos t, \sin t)$ in $\mathbb{S}^2(2)$ with the round metric.
4. $[0, 1] \longrightarrow \mathbb{U}^2(1),\ t \longmapsto (t, t+1)$ in the Poincaré halfspace model.
5. $[0, 1] \longrightarrow \mathbb{U}^2(2),\ t \longmapsto (t, t+1)$ in the Poincaré halfspace model.
6. $[0, 2\pi] \longrightarrow \mathbb{U}^2(1),\ t \longmapsto (\cos t, \sin t + 2)$ in the Poincaré halfspace model.

Exercises (for the session on December 7/8)**.** The following exercises (which all are solvable with the material read/discussed in week 4) will be discussed.

Please turn over

Exercise 4.1 (scaled models)**.** Let $n \in \mathbb{N}$ and $R \in \mathbb{R}_{>0}$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. There exists an isometry $(\mathbb{S}^n(R), g_R) \longrightarrow (\mathbb{S}^n(1), R^2 \cdot g_1)$, where g_R denotes the round metric on $\mathbb{S}^n(R)$.
2. There exists an isometry $(\mathbb{H}^n(R), g_R) \longrightarrow (\mathbb{H}^n(1), R^2 \cdot g_1)$, where g_R denotes the standard hyperbolic Riemannian metric on $\mathbb{H}^n(R)$.

Exercise 4.2 (the Möbius transformation action on the hyperbolic plane)**.** We consider the Möbius transformation action

$$\begin{aligned} \mathrm{SL}(2,\mathbb{R}) \times \mathbb{U}^2(1) &\longrightarrow \mathbb{U}^2(1) \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) &\longmapsto \frac{a \cdot z + b}{c \cdot z + d} \end{aligned}$$

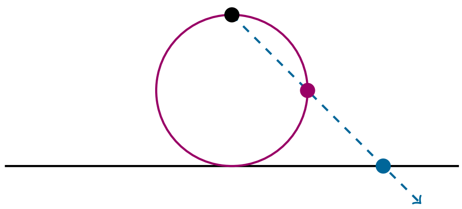
on the Poincaré halfspace model $\mathbb{U}^2(1)$ (viewed as a subset of $\mathbb{C}$). Solve two of the following problems:

1. Show that this defines an action on $\mathbb{U}^2(1)$.
2. Show that this action is isometric with respect to $g^{\mathbb{U}}$.
3. Is this action transitive? Justify your answer!
4. What is the stabiliser group at i ?

Exercise 4.3 (spheres are locally conformally flat)**.** Let $n \in \mathbb{N}$ and let g be the Euclidean Riemannian metric on $\mathbb{R}^n$. Show that the round n-sphere $(\mathbb{S}^n, g^\circ)$ is *locally conformally flat*, i.e., that around each point of $\mathbb{S}^n$, there exists a smooth chart $\varphi\colon U \longrightarrow U'$ and a smooth function $f\colon U' \longrightarrow \mathbb{R}_{>0}$ with

$$(\varphi^{-1})^* g^\circ = f^2 \cdot g.$$

Hints. Use homogeneity to reduce the problem to the south pole. Then use the spherical stereographic projection and proceed as in the comparison between the hyperboloid model and the Poincaré disk model. In fact, the formulas will be very similar (just some signs will flip).



Exercise 4.4 (Cayley transform)**.** Let $n \in \mathbb{N}_{>0}$. Show that the Cayley transform

$$\begin{aligned} c\colon\ \mathbb{U}^n(R) &\longrightarrow \mathbb{B}^n(R) \\ \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \ni (x,y) &\longmapsto \frac{1}{\|x\|_2{}^2 + (y+R)^2} \cdot \bigl(2 \cdot R^2 \cdot x, R \cdot (\|x\|_2{}^2 + \|y\|_2{}^2 - R^2)\bigr) \end{aligned}$$

is an isometry $(\mathbb{U}^n(R), g^{\mathbb{U}}) \longrightarrow (\mathbb{B}^n(R), g^{\mathbb{B}})$.
Hints. In order to manage the complexity of this task, you may restrict to the case of $n = 2$ (without using complex analysis, because this would not generalise directly to the higher-dimensional case) or resort to the help of computer algebra systems (but your submission has to be comprehensible and verifiable for a human reader).

Bonus problem (hyperbolic art)**.** Sketch how Escher's *Cirkellimiet IV* would look like in the Poincaré halfplane model!

Submission before December 3, 2020, 10:00, via email to your tutor.