Differential Geometry I: Week 6

Prof. Dr. C. Löh/AG Ammann December 3, 2020

Reading assignment (for the lecture on December 9)**.** In addition to lengths, we can also consider volumes. Moreover, we will take the first step towards the introduction of analytic notions of curvature.

- Read Chapter 1.5.3 *Volume and orientation.*

  We will use volumes only sparsely in this course. This is why the treatment is mostly just a short recap of material that most of you might have seen before. If you don't know anything about differential forms, this will probably not be much of a problem for the rest of the course.

- Read Chapter 2.1 *The idea of curvature.*

Reading assignment (for the lecture on December 10)**.** We now formalise the notion of connections. Connections will be extremely important for the rest of this course; thus, it is best to become good friends with them as soon as possible!

- Read Chapter 2.2.1 *Connections.*
- Read Chapter 2.2.2 *Local descriptions of connections.*

Next week, we will apply connections to vector fields along curves, introduce geodesics and parallel transport. Finally, we will add compatibility with Riemannian metrics to the picture.

Étude (the Euclidean connection)**.** Let $\overline{\nabla}$ denote the Euclidean connection on $\mathbb{R}^2$, let $X := ((x,y) \mapsto (1,0))$, let $Y := ((x,y) \mapsto (0,1))$, and let $Z := ((x,y) \mapsto (x^2, x \cdot y^3))$.

1. Show the property (FL1) for $\overline{\nabla}$.
2. Show the property (L2) for $\overline{\nabla}$.
3. Show the property (F2) for $\overline{\nabla}$.
4. Compute $\overline{\nabla}_X Y$ and $\overline{\nabla}_X X$.
5. Compute $\overline{\nabla}_X Z$ and $\overline{\nabla}_Z X$.
6. Compute $\overline{\nabla}_Z Z$.

Exercises (for the session on December 14/15)**.** The following exercises (which all are solvable with the material read/discussed in week 5) will be discussed.

Please turn over

Exercise 5.1 (Heisenberg group)**.** We consider the subgroup (*Heisenberg group*)

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\} \subset \mathrm{SL}(3, \mathbb{Z})$$

of $\mathrm{SL}(3, \mathbb{Z})$ and its action by matrix multiplication on $\mathbb{R}^3$. Which of the following statements are true? Justify your answer!

1. The action is isometric with respect to the Euclidean Riemannian metric.
2. The action is free and proper.

Exercise 5.2 (Klein bottle)**.** We consider the following action

$$\begin{aligned} (\mathbb{Z} \rtimes \mathbb{Z}) \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \bigl((n_1, n_2), (x_1, x_2)\bigr) &\longmapsto \bigl((-1)^{n_2} \cdot x_1 + n_1, x_2 + n_2\bigr) \end{aligned}$$

on $\mathbb{R}^2$, where $\mathbb{Z} \rtimes \mathbb{Z}$ is defined via the action of $1 \in \mathbb{Z}$ by multiplication by -1 on $\mathbb{Z}$.

1. Show that this action is isometric (with respect to the Euclidean Riemannian metric), proper, and free. The Riemannian quotient of this action is the *Klein bottle*.
2. Show that there exists an action of $\mathbb{Z}/2$ on a suitably scaled 2-torus T such that the quotient $(\mathbb{Z}/2) \setminus T$ with the quotient Riemannian metric is isometric to the Klein bottle.

Exercise 5.3 (the punctured plane)**.** We consider $M := \mathbb{R}^2 \setminus \{0\}$ with the smooth structure induced by $\mathbb{R}^2$ and the Euclidean Riemannian metric g. Let $x := (1, 0)$, $y := (-1, 0) \in M$.

1. Compute $d_g(x, y)$! Justify your answer!
2. Is there a piecewise regular curve $\gamma \colon [0, 1] \longrightarrow M$ with $\gamma(0) = x$ and $\gamma(1) = y$ as well as $L_g(\gamma) = d_g(x, y)$? Justify your answer in detail and illustrate your arguments with suitable pictures!

Exercise 5.4 (quotients of symmetric spaces)**.** Let (M, g) be a symmetric space and let $\Gamma \curvearrowright (M, g)$ be an isometric, free, and proper action of a discrete group Γ. Show that the quotient $\Gamma \setminus M$ (with the induced smooth structure and Riemannian metric) is a locally symmetric space.
Hints. Use the Riemannian distance function to find invariant neighbourhoods in the quotient.

Bonus problem (locally symmetric spaces from lattices)**.**

1. Look up the terms *Lie group* and *lattice in a Lie group* in the literature.
2. How do lattices in Lie groups lead to locally symmetric spaces?

   Hints. Giving a precise formulation of such a statement (and a reference) and briefly explaining the terms/construction is enough.

Submission before December 10, 2020, 10:00, via email to your tutor.