Differential Geometry I: Week 7

Prof. Dr. C. Löh/AG Ammann December 10, 2020

Reading assignment (for the lecture on December 16). Using linear connections, we can differentiate vector fields along curves. In particular, this allows to introduce an analytic notion of geodesics.

- Read Chapter 2.2.3 *Covariant derivatives along curves.*
- Recall basics on ordinary differential equations (in particular: existence and uniqueness theorems for solutions of ordinary differential equations).
- Read Chapter 2.2.4 *Geodesics* until Theorem 2.2.22.

Reading assignment (for the lecture on December 17). Covariant derivatives along curves give also rise to parallel transport of tangent vectors along curves. This parallel transport provides a nice, geometric, reinterpretation of covariant derivatives and connections.

- Read the rest of Chapter 2.2.4 *Geodesics.*
- Read Chapter 2.2.5 *Parallel transport.*
- Read the introduction of Chapter 2.3 *The Levi-Civita connection.*

Étude (covariant derivatives along curves). Let

$$\begin{aligned} \gamma \colon \mathbb{R} &\longrightarrow \mathbb{R}^2, & t &\longmapsto (\cos t, \sin t) \\ X \colon \mathbb{R} &\longrightarrow \mathbb{R}^2, & t &\longmapsto (-t \cdot \sin t, t \cdot \cos t) \\ Y \colon \mathbb{R} &\longrightarrow \mathbb{R}^2, & t &\longmapsto (42, 0) \\ Z \colon \mathbb{R} &\longrightarrow \mathbb{R}^2, & t &\longmapsto (t, t^{2020}). \end{aligned}$$

Which of the terms

$$D_\gamma X, \qquad D_\gamma Y, \qquad D_\gamma Z$$

make sense

- with respect to the Euclidean linear connection on $\mathbb{R}^2$?
- with respect to the linear connection on $\mathbb{S}^1$ induced by the Euclidean linear connection on $\mathbb{R}^2$?

In case they do make sense, compute them explicitly.

Exercises (for the session on December 21/22). The following exercises (which all are solvable with the material read/discussed in week 6) will be discussed.

Please turn over

Exercise 6.1 (volumes). Let M be a compact non-empty smooth manifold, let g_1, g_2 be Riemannian metrics on M, and let $\lambda \in \mathbb{R}_{>0}$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

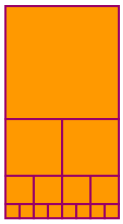
1. If $g_2 = \lambda^2 \cdot g_1$, then $\operatorname{vol}(M, g_2) = \lambda^n \cdot \operatorname{vol}(M, g_1)$.
2. If $\operatorname{vol}(M, g_2) = \lambda^n \cdot \operatorname{vol}(M, g_1)$, then $g_2 = \lambda^2 \cdot g_1$.

Exercise 6.2 (volume growth of hyperbolic spaces). Let $n \in \mathbb{N}_{\geq 2}$. We equip the halfspace $\mathbb{U}^n := \mathbb{U}^n(1)$ with the metric $g^{\mathbb{U}}$ (of radius 1). Show that there exist constants $a \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}_{>0}$ such that for each $x \in \mathbb{U}^n$, we have

$$\forall_{r \in \mathbb{R}_{>0}} \quad \varrho_x^{(\mathbb{U}^n, g^{\mathbb{U}})}(r) \geq c \cdot r^{n-1} \cdot a^r.$$

Hints. Look at the point $(0, \dots, 0, 1)$ and sets of the form $\{(x_1, \dots, x_{n-1}, y) \in \mathbb{U}^n \mid y \in [e^{-r/(2n)}, 1],\ x_1, \dots, x_{n-1} \in [0, r/(2n)]\}$; then estimate the Riemannian distance function and the Riemannian volume in the correct direction.

Bonus problem. How is this related to the picture below?



Exercise 6.3 (construction of connections). Solve two of the following problems:

1. Show that the pullback of a linear connection is a linear connection (Proposition 2.2.4).
2. Show that the induced connection on submanifolds is well-defined (Example 2.2.8).
3. Show that the induced connection on submanifolds is a connection (Example 2.2.8).
4. Show Proposition 2.2.11 on restrictions of connections.

Exercise 6.4 (connection coefficients). Compute the connection coefficients of the Euclidean linear connection on $\mathbb{R}^2 \setminus \{0\}$ with respect to polar coordinates.
Hints. This requires some careful bookkeeping ... Don't panic!

Bonus problem (the affine space of connections). Let M be a smooth manifold and let ∇ be a linear connection on M. Show that the set of all linear connections on M can alternatively be described as

$$\big\{\text{“}\nabla + D\text{”} \bigm| D \in \Gamma(\mathbf{T}^{(1,2)}\, M)\big\}.$$

Hints. You should first give a sensible interpretation for the term “$\nabla + D$”.

Submission before December 17, 2020, 10:00, via email to your tutor.