Differential Geometry I: Week 9

Prof. Dr. C. Löh/AG Ammann January 7, 2021

Reading assignment (for the lecture on January 13). We continue the study of the Riemannian curvature tensor. In particular, we characterise flat manifolds.

- Read the rest of Chapter 2.4.1 *The Riemannian curvature tensor.*
- Read Chapter 2.4.2 *Flat manifolds.*

Reading assignment (for the lecture on January 14). From the Riemannian curvature tensor, we extract various degrees of information. Moreover, we start with the computation of the curvatures of model spaces.

- Read Chapter 2.4.3 *Sectional curvature.*
- Read Chapter 2.4.4 *Ricci curvature.*
- Read Chapter 2.4.5 *Scalar curvature.*
- Read Chapter 2.5.1 *Locally conformally flat manifolds.*

Next week, we will complete the curvature computations for model spaces and we will start working with Riemannian geodesics.

Étude (tensors). Let M be a smooth manifold and let ∇ be a linear connection on M. Which of the following terms define tensor fields on M ? Here, $X, Y, \dots$ denote vector fields on M.

1. $X \longmapsto \nabla_X X$
2. $(X, Y) \longmapsto \nabla_X Y$
3. $(X, Y) \longmapsto \nabla_X Y - \nabla_Y X$
4. $(X, Y) \longmapsto [X, Y]$
5. $(X, Y, Z) \longmapsto \nabla_{[X,Y]} Z$
6. $(X, Y, Z) \longmapsto \nabla_{[Y,Z]} X + \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X$

Exercises (for the session on January 18/19). The following exercises (which all are solvable with the material read/discussed in week 8) will be discussed.

Please turn over

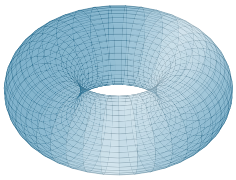
Exercise 8.1 (scaled manifolds)**.** Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let $\lambda \in \mathbb{R}_{>0}$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. The Levi-Civita connection of $(M, \lambda \cdot g)$ is $\lambda \cdot \nabla$.
2. The Riemannian curvature $(3, 1)$-tensor of $(M, \lambda \cdot g)$ coincides with the Riemannian curvature $(3, 1)$-tensor of (M, g).

Exercise 8.2 (the return of the torus)**.** We consider the 2-torus $T \subset \mathbb{R}^3$, given by

$$\Big\{ \big((2 + \cos x) \cdot \cos y, (2 + \cos x) \cdot \sin y, \sin x \big) \;\Big|\; x, y \in \mathbb{R} \Big\} \subset \mathbb{R}^3$$

and equipped with the Riemannian metric induced by the Euclidean Riemannian metric on $\mathbb{R}^3$. Show that the Riemannian curvature $(3, 1)$-tensor of T is *not* everywhere zero.



Exercise 8.3 (tensor characterisation lemma; Proposition 2.3.2)**.** Prove the tensor characterisation lemma.

Exercise 8.4 (compatible connections; Proposition 2.3.7)**.** Let (M, g) be a Riemannian manifold, let ∇ be a linear connection on M and suppose that the parallel transport maps with respect to ∇ are isometries. Show that for all smooth curves γ on M and all $X, Y \in \Gamma(\mathrm{T}\, M|\gamma)$, we have

$$\langle X, Y \rangle_g' = \langle D_\gamma X, Y \rangle_g + \langle X, D_\gamma Y \rangle_g.$$

Hints. Show that parallel orthonormal frames exist and use them.

Bonus problem (commuting vector fields)**.**

1. What is the flow of a vector field? What can you say about existence of flows of vector fields?
2. Prove the following realisation theorem on commuting vector fields: Let M be a smooth manifold of dimension n and let $E_1, \dots, E_n \in \Gamma(\mathrm{T}\, M)$ be smooth vector fields on M with the following properties:

   - The family $(E_1, \dots, E_n)$ is a frame.
   - For all $j, k \in \{1, \dots, n\}$, we have $[E_j, E_k] = 0$.

   Show that for each $x \in M$, there exists an open neighbourhood U around x such that $(E_j|_U)_{j \in \{1,\dots,n\}}$ is a *coordinate* frame on U.

   Hints. Use iterated flows to find a suitable chart.

Submission before January 14, 2021, 10:00, via email to your tutor.