17.06.2020

Recap: graphings of standard eq rels
(measured version of gen sets)

$\boxed{B.2.2}$ COST OF MEASURED EQUIVALENCE RELATIONS

Definition. (cost of a graphing/measured eq rel).
Let $(\mathcal{R}, \mu)$ be a measured eq rel.

- The <u>cost</u> of a graphing $\Phi = (\underbrace{\varphi_i}_{: A_i \to B_i})_{i \in I}$ of $\mathcal{R}$ is

measured version of the number of elts of Φ

$$\mathrm{cost}_\mu \Phi := \sum_{i \in I} \underbrace{\mu(A_i)}_{= \mu(B_i) = \frac{1}{2} \cdot (\mu(A_i) + \mu(B_i))} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

- The <u>cost</u> of $\mathcal{R}$ is "minimal number of generators"

$$\mathrm{cost}_\mu \mathcal{R} := \overset{\inf}{\cancel{\min}} \{ \mathrm{cost}_\mu \Phi \mid \Phi \text{ is a graphing of } \mathcal{R} \}$$
$$\in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

⚠ In gen, the inf is <u>not</u> attained!

Example. Let X be a standard Borel space and let μ be a measure on X.
Then

$$\mathrm{cost}_\mu \underbrace{\Delta_X}_{= \{(x,x) \mid x \in X\}} = 0. \quad \text{(use the empty graphing)}$$

Example. Let $\Gamma \curvearrowright (X,\mu)$ be a standard probability action. Then

$$\operatorname{cost}_\mu \mathcal{R}_{\Gamma \curvearrowright X} \leq \underbrace{d(\Gamma)}$$

$:= \min\{|S| \mid S \subset \Gamma \text{ generates } \Gamma\}$
$\in \mathbb{N} \cup \{\infty\}$

wlog $S \subset \Gamma$ is a fin. gen set

$\leadsto \Phi_S := \begin{pmatrix} X \to X \\ x \mapsto \gamma \cdot x \end{pmatrix}_{\gamma \in S}$ is a graphing of $\mathcal{R}_{\Gamma \curvearrowright X}$

$$\leadsto \operatorname{cost}_\mu \mathcal{R}_{\Gamma \curvearrowright X} \leq \operatorname{cost}_\mu \Phi_S$$
$$= \sum_{\gamma \in S} \underbrace{\mu(X)}_{=1} = |S|.$$

Concrete example:

$$\operatorname{cost}_\lambda \mathcal{R}_{\underbrace{R_\alpha}_{\in \mathbb{R}} : \mathbb{Z} \curvearrowright S^1} \leq 1.$$

- Realizing $\mathcal{R}_{\Gamma \curvearrowright X}$ as the orbit relation of an action of a different group might lead to a different bound
- In general: the above estimate is very bad!

Goal: compute cost in many examples (and apply it to OE problems)

3.2.3 BASIC COST ESTIMATES

Proposition. (cost is only attained by treeings). Let $(\mathcal{R}, \mu)$ be a measured eq rel with finite cost and let Φ be a graphing of $\mathcal{R}$ that is <u>not</u> a treeing (up to μ-null sets). Then there ex. a graphing Φ' of $\mathcal{R}$ with

$$\mathrm{cost}_\mu \Phi' < \mathrm{cost}_\mu \Phi.$$

In particular: If Φ is a graphing of $\mathcal{R}$ with $\mathrm{cost}_\mu \Phi = \mathrm{cost}_\mu \mathcal{R}$, then (up to μ-null sets) Φ is a (reduced) treeing of $\mathcal{R}$.

Sketch of proof. Wlog let $\mathrm{cost}_\mu \Phi < \infty$.

As Φ is <u>not</u> a treeing, then ex. a loop in Φ (on a set A of positive measure)

$\leadsto$ φ is redundant (can replace it by the rest of the loop)

Problem: φ might appear in the rest of the loop (on elts that we cannot directly)

$\leadsto$ change the loop to shrink A appropriately. $\square$

Proposition. (cost of sections). Let $(\mathcal{R},\mu)$ be measured eq rel on X and let $A \subset X$ be a section of $\mathcal{R}$. Then:

measurable subset that meets every orbit

$$\mathrm{cost}_\mu \mathcal{R} = \mathrm{cost}_{\mu|_A} \mathcal{R}|_A + \mu(X \setminus A).$$

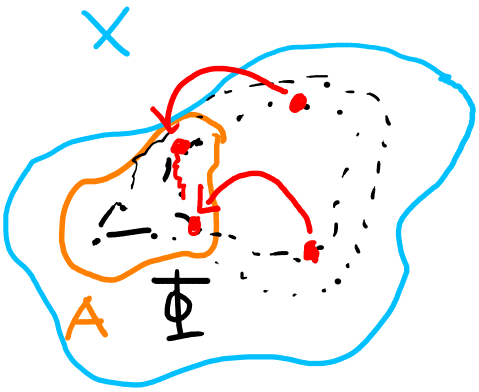
Reformulation: If μ is a prob. measure on X:

$$\frac{1}{\mu(A)} \cdot \left(\mathrm{cost}_\mu \mathcal{R} - 1\right) = \mathrm{cost}_{\frac{1}{\mu(A)}\cdot\mu|_A} \mathcal{R}|_A - 1.$$

(looks like Nielsen–Schreier thm!)

Sketch of proof.

- $\leq$: let Φ be a graphing of $\mathcal{R}|_A$.



Only need a way to hop from $X\setminus A$ to A

(need to do this in a measured way)

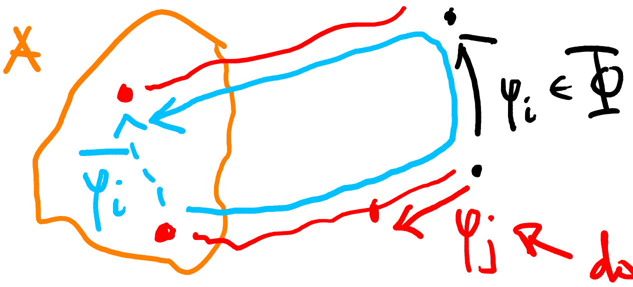
(one can use the Feldman–Moore thm to organize this) $\leadsto$ additional cost: $\mu(X\setminus A)$

$$\leadsto \mathrm{cost}_\mu \mathcal{R} \leq \mathrm{cost}_{\mu|_A} \Phi + \mu(X\setminus A).$$

- $\geq$: let Φ be a graphing of $\mathcal{R}$.

$\leadsto (\overline{\varphi_i})_{i\in I}$ graphing of $\mathcal{R}|_A$ with cost $\leq \mathrm{cost}_\mu \Phi$

How to save $\mu(X\setminus A)$?



φ_j don't need $\overline{\varphi_j}$

$\square$

Corollary. (Lower cost bound for aperiodic relations). Let $(\mathcal{R}, \mu)$ be an aperiodic measured eq rel on X with $\mu(X) < \infty$.

all orbits are infinite

Then

$$\mathrm{cost}_\mu \mathcal{R} \geq \mu(X).$$

(In general, does not hold without the aperiodicity condition: e.g., Δ_X)

Proof. By the marker lemma: there ex. a vanishing seq $(A_n)_{n \in \mathbb{N}}$ of markers for $\mathcal{R}$

$\leadsto$ sections for $\mathcal{R}$

$\leadsto A_0 \supset A_1 \supset \dots$

and $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$

$\leadsto \lim_{n \to \infty} \mu(A_n) = 0$

$\mu(X) < \infty$

Then:

$$\forall_{n \in \mathbb{N}} \ \mathrm{cost}_\mu \mathcal{R} = \underbrace{\mathrm{cost}_{\mu|_{A_n}} \mathcal{R}|_{A_n}}_{\geq 0} + \mu(X \setminus A_n)$$

$$\geq \mu(X \setminus A_n)$$

$$= \mu(X) - \underbrace{\mu(A_n)}_{\xrightarrow[n \to \infty]{} 0}$$

$$\leadsto \mathrm{cost}_\mu(\mathcal{R}) \geq \mu(X). \qquad \square$$