Ergodic Theory of Groups: Week 10

Prof. Dr. C. Löh/J. Witzig June 17, 2020

Reading assignment (for the lecture on June 23). This week, we will mainly be concerned with the computation of cost of (free) products and its application to orbit equivalence rigidity.

- Read the rest of Chapter 3.2.3 *Basic cost estimates*.
- (Optional) Recall the notion of *(amalgamated) free products* of groups.
- Read Chapter 3.2.4 *Cost of free products* with the exception of the proof of the decomposition lemma (Lemma 3.2.29; it has to wait for the next lecture!).

  The proofs might look technical and involved at first; however, the underlying geometric ideas are very nice and worth being studied.

Reading assignment (for the lecture on June 24).

- Read the rest of Chapter 3.2.4 *Cost of free products*, i.e., read the proof of the decomposition lemma (Lemma 3.2.29).
- Read Chapter 3.2.5 *Application: Rigidity of free groups*.

  All the hard work is done by now – we just need to use the results!
- Recall the notions of *rank gradient, residual chains, and residual finiteness*.
- Read Chapter 3.2.6 *Cost of products*.

  This is much easier than the case of free products!

Next week, we will introduce cost of groups and, in particular, we will use cost to compute rank gradients of groups.

Implementation (generating sets of products). Read the second Isabelle fragment: *Generating sets of products* (Appendix A.4); you might also want to interact with it:

http://www.mathematik.uni-r.de/loeh/teaching/erg_ss2020/2306/Product_GenSet.thy

This fragment uses functionality of the Isabelle library HOL-Algebra (the definition of groups and of generating sets).

We might discuss this fragment in one of the lectures.

Exercises (for the session on June 26). The following exercises (which all are solvable with the material read/discussed in week 9) will be discussed.

Please turn over

Exercise 9.1 (graphings). Let $\mathcal{R}$ be a standard equivalence relation on a standard Borel space X and let $A \subset X$ be measurable. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $\Phi = (\varphi_i)_{i \in I}$ is a graphing of $\mathcal{R}$, then $(\varphi_i|_A)_{i \in I}$ is a graphing of $\mathcal{R}|_A$.
2. If Φ is a graphing of $\mathcal{R}|_A$ and Θ is a graphing of $\mathcal{R}_{X \setminus A}$, then $\Phi \cup \Theta$ is a graphing of $\mathcal{R}$.

Exercise 9.2 (treeings of smooth equivalence relations). Let $(\mathcal{R}, \mu)$ be a smooth measured equivalence relation on a standard Borel space X with fundamental domain A and let $\mu(X) < \infty$.

1. Show that the relation $\mathcal{R}$ admits a treeing.
2. Show that if Ψ is a reduced treeing of $\mathcal{R}$, then $\operatorname{cost}_\mu \Psi = \operatorname{cost}_\mu \mathcal{R}$.

Exercise 9.3 (Poincaré recurrence for measured equivalence relations).

1. Formulate a version of the Poincaré recurrence theorem for measured equivalence relations.
2. Prove this version of Poincaré recurrence.

   Hints. Feldman-Moore theorem?!

Exercise 9.4 (implementation: orbit relations). We consider the Isabelle library HOL-Algebra.Group_Action:

https://isabelle.in.tum.de/library/HOL/HOL-Algebra/document.pdf

In particular, this library contains a proof that orbit relations of group actions are equivalence relations.

1. Which lemmas of this library combine to give this result?
2. Convert the proof of symmetry given in this library into a pen-and-paper proof (including the necessary definitions of group actions, orbits, etc.).

Hints. The *carrier* is just the underlying set of a *monoid_scheme*. This is one of the inconvenient artefacts introduced through the interaction between typed set theory and type theory in Isabelle.

Bonus problem (formal proofs II).

1. What does the *four colour theorem* say?
2. Give the reference for the first complete proof of the four colour theorem.
3. Give a (the?) reference for a proof of the four colour theorem in a proof assistant? Which proof assistant was used?
4. Also the first proof of the four colour theorem made use of computer support (and thus was heavily disputed). What is the principal difference between this sort of computer support and a formalised proof in a proof assistant?

Submission before June 24, 8:00, via email to johannes.witzig@ur.de or through git.