Ergodic Theory of Groups: Week 13

Prof. Dr. C. Löh/J. Witzig July 8, 2020

Reading assignment (for the lecture on July 14)**.** This week, we will establish Dye's theorem on the orbit equivalence of all ergodic atom-free standard probability actions of $\mathbb{Z}$.

- Read Chapter 4.2.2 *The Rokhlin lemma*
- Read Chapter 4.2.3 *Dye's theorem* until Lemma 4.3.17.

Reading assignment (for the lecture on July 15)**.**

- Read the rest of Chapter 4.2.3 *Dye's theorem*.
- Read Chapter 4.3.1 *Hyperfiniteness*.

Next week will be the final week of this course. We will complete our treatment of general amenable groups. In the very last lecture, I will explain how I ended of in this type of ergodic theory, i.e., how it helped to solve problems on integral approximation of simplicial volume.

Implementation (Følner sequences)**.** Read the fifth Isabelle fragment *Følner sequences* (Appendix A.4) on Følner sequences and an example of a Følner sequence in the additive group $\mathbb{Z}$; you might also want to interact with it:

http://www.mathematik.uni-r.de/loeh/teaching/erg_ss2020/1407/Folner.thy

This fragment uses functionality of the Isabelle library HOL-Algebra (the definition of groups) and of the library HOL-Analysis (for convergence/limits of sequences). Moreover, the notation from Set_Intervals is used. We might discuss this fragment in one of the lectures.

Exercises (for the session on July 17)**.** The following exercises (which all are solvable with the material read/discussed in week 12) will be discussed.

Please turn over

Exercise 12.1 (amenability and restrictions)**.** Let $(\mathcal{R}, \mu)$ be a measured equivalence relation on a standard Borel probability space, let $A \subset X$ with $\mu(A) > 0$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $\mathcal{R}|_A$ is $\mu|_A$-amenable, then $\mathcal{R}$ is μ-amenable.
2. If $\mathcal{R}$ is μ-amenable, then $\mathcal{R}|_A$ is $\mu|_A$-amenable.

Exercise 12.2 (hyperfiniteness and almost invariance)**.** Let (X, μ) be a standard Borel probability space, let $\Gamma \curvearrowright X$ be an action by a countable group Γ by measurable isomorphisms, and let $\mathcal{R} := \mathcal{R}_{\Gamma \curvearrowright X}$. Show that the following conditions are equivalent:

1. The equivalence relation $\mathcal{R}$ is μ-hyperfinite.
2. For every $\varepsilon \in \mathbb{R}_{>0}$ and every finite subset $S \subset \Gamma$, there is a finite measurable equivalence relation $\mathcal{R}' \subset \mathcal{R}$ with
$$\mu\bigl(\bigl\{x \in X \bigm| \forall_{\gamma \in S} \quad (\gamma \cdot x, x) \in \mathcal{R}'\bigr\}\bigr) > 1 - \varepsilon.$$

Hints. One implication is very straightforward (which one?!). For the other one, enumerate the group Γ and look at finite initial sets of this enumeration. When in trouble, approximate with exponential accuracy.

Exercise 12.3 (cost of hyperfinite equivalence relations)**.** Let $(\mathcal{R}, \mu)$ be a measured equivalence relation on a standard Borel probability space. Show that if $\mathcal{R}$ is μ-hyperfinite, then
$$\operatorname{cost}_\mu \mathcal{R} \leq 1.$$

Exercise 12.4 (implementation: Følner sequences for finite groups)**.** We consider the file

http://www.mathematik.uni-r.de/loeh/teaching/erg_ss2020/0807/Folner_Finite_Exercise.thy

This depends on Folner.thy. Complete this file, i.e.:

1. Give a pen-and-paper proof of the fact that finite groups admit Følner sequences. Split up the proof already in intermediate steps that are suitable for formalisation/implementation.
2. Implementation: Fill in the proof of *finite_groups_admit_Folner_seq*.

Bonus problem (arXiv)**.**

1. Look at today's new announcements of math preprints at the preprint archive *arXiv*: https://arxiv.org/list/math/new .
2. Find at least one new preprint that is related to ergodic theory (of groups).
3. (Without submission) Improve your skills at: http://snarxiv.org/vs-arxiv/

Submission before July 15, 8:00, via email to johannes.witzig@ur.de or through git.