Ergodic Theory of Groups: Week 2

Prof. Dr. C. Löh/J. Witzig April 22, 2020

Reading assignment (for the lecture on April 28)**.** In the next few lectures, we will collect and construct a list of examples of dynamical systems that will accompany us for the rest of this course. Therefore, we will spend some time to carefully develop and understand these examples.

- Read Chapter 1.2.1 *Rotations and shifts on the circle.*
- Read Chapter 1.2.2 *Coset translations* up to Quick check 1.2.11.
- Did you do all the quick checks, try out the interactive tool, check all the arguments, understand the underlying ideas, and write down the questions you have for the interactive discussion sessions? (Clearly, these questions will be relevant for every lecture.)

Reading assignment (for the lecture on April 29)**.**

- Read the rest of Chapter 1.2.2 *Coset translations* (starting from Definition 1.2.12).

  Lattices show up in all kinds of applications. If you are not familiar with topological groups, Haar measures, etc.: Don't worry, we will not go deep into this theory; we will just use lattices as a nice source of examples (and maybe at the very end of the course for various applications).

  If you want to learn more about the ambient groups for lattices, you might be interested in following the course *Algebraische Gruppen* (taught by V. Ertl and D. Schäppi).

- Read Chapter 1.2.3 *Diagonal actions.*

  If necessary: First recall product probability spaces (e.g., as in Appendix A.1.4 and Exercise 0.3).

- Did you save the princess?

Exercises (for the session on "May 1")**.** The exercise session will be moved to a non-holiday. The following exercises (which all are solvable with the material read/discussed in week 1) will be discussed.

Please turn over

Exercise 1.1 (the real plane)**.** We consider the matrices

$$A := \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 3 & 3 \\ 4 & 2 \end{pmatrix}$$

in $M_{2\times 2}(\mathbb{R})$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. The map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $x \longmapsto A \cdot x$ is measure preserving (with respect to the Lebesgue measure on $\mathbb{R}^2$).

2. The map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $x \longmapsto B \cdot x$ is measure preserving (with respect to the Lebesgue measure on $\mathbb{R}^2$).

Exercise 1.2 (isometries on the 3-ball)**.** Let $\Gamma \subset \mathrm{SO}(3)$ be a countable subgroup and let $D^3 \subset \mathbb{R}$ be the Euclidean unit 3-ball. Show that the (well-defined!) action

$$\begin{aligned} \Gamma \times D^3 &\longrightarrow D^3 \\ (A, x) &\longmapsto A \cdot x \end{aligned}$$

given by matrix multiplication is measure preserving with respect to the Lebesgue measure and essentially free.
Hints. What do you know about elements of $\mathrm{SO}(3)$?

Exercise 1.3 (the non-free locus)**.** Let X be a Hausdorff topological space, let μ be a measure on the Borel σ-algebra of X, and let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action. Show that the set $A := \{x \in X \mid \Gamma_x \not\cong 1\}$. is measurable and that $\Gamma \cdot A = A$.

Exercise 1.4 (a free action without measurable fundamental domain)**.** We consider the action of $\mathbb{Q}$ on $\mathbb{R}$ given by addition of real numbers; this action is measure preserving with respect to the Lebesgue measure on $\mathbb{R}$. Show that there is *no* measurable set $D \subset \mathbb{R}$ that meets every $\mathbb{Q}$-orbit exactly once, i.e., that satisfies

$$\forall_{x \in \mathbb{R}} \quad \bigl|(\mathbb{Q} + x) \cap D\bigr| = 1.$$

Bonus problem (initial/terminal objects)**.**

1. Does the category $\mathsf{Meas_p}$ of measure spaces and measure preserving maps contain an initial object?

2. Does the category $\mathsf{Meas_p}$ contain a terminal object?

3. Does the category Measbl of measurable sets and measurable maps contain an initial object?

4. Does Measbl contain a terminal object?

Submission before April 29, 2020, 8:00, via email to johannes.witzig@ur.de or through git. Solutions should be in PDF format and may be submitted in English or German.