Ergodic Theory of Groups: Week 3

Prof. Dr. C. Löh/J. Witzig April 29, 2020

Reading assignment (for the lecture on May 5)**.** We continue with the construction of examples of dynamical systems.

- Recall the construction of the p-adic integers (including both a concrete description and the notion of inverse limits).
- Read Chapter 1.2.4 *Profinite completions.*

  Profinite completions are somewhat hard to visualise and to get used to. Don't worry too much if you think that this is difficult.
- Read Example 1.2.41 *Bernoulli shift.*

Reading assignment (for the lecture on May 6)**.** We study the last class of examples (for now) and engage in a first, quick excursion into long-term behaviour of dynamical systems.

- Complete reading Chapter 1.2.5 *Bernoulli shifts.*
- Read Chapter 1.3 *Recurrence.*

  In case you don't know much about functional analysis, feel free to ignore the technical details of the limit argument of the derivation of Szemerédi's theorem from the multiple recurrence theorem. However, you should at least extract the structure of the main line of argument.

Next week, we will introduce different ways of comparing different dynamical systems.

Exercises (for the session on May 8)**.** The following exercises (which all are solvable with the material read/discussed in week 2) will be discussed.

Please turn over

Exercise 2.1 (diagonal). Let Γ be a countable group and let $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ be probability measure preserving actions. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If the diagonal action $\Gamma \curvearrowright (X \times Y, \mu \otimes \nu)$ is essentially free, then $\Gamma \curvearrowright (X, \mu)$ is essentially free or $\Gamma \curvearrowright (Y, \nu)$ is essentially free.
2. If all orbits of $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are finite, then also all orbits of the diagonal action $\Gamma \curvearrowright (X \times Y, \mu \otimes \nu)$ are finite.

Exercise 2.2 (Heisenberg group). We consider the integral and the real *Heisenberg group* (which both are subgroups of $\mathrm{SL}(3, \mathbb{R})$):

$$H_{\mathbb{Z}} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\}, \qquad H_{\mathbb{R}} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

Show that $H_{\mathbb{Z}}$ is a cocompact lattice in $H_{\mathbb{R}}$ with respect to the topology given by convergence of all matrix coefficients.

Exercise 2.3 (discrete subgroups). Let G be a locally compact second countable topological group and let $\Lambda \subset G$ be a discrete subgroup (i.e., Λ is a subgroup of G and the subspace topology on Λ is the discrete topology). You only need to submit a solution to one of the following:

1. Show that Λ is countable.
2. Show that the left translation action of Λ on G admits a *measurable fundamental domain*, i.e., that there exists a measurable subset $D \subset G$ with

$$\forall_{x \in G} \quad |\Gamma \cdot x \cap D| = 1.$$

Exercise 2.4 (finite index (normal) subgroups). Let Γ be a group. You only need to submit a solution to one of the following:

1. Let $\Lambda_1, \Lambda_2 \subset \Gamma$ be subgroups of finite index. Show that also $\Lambda_1 \cap \Lambda_2$ has finite index in Γ.
2. Let $\Lambda \subset \Gamma$ be a finite index subgroup. Show that there exists a normal subgroup $N \subset \Gamma$ of finite index with $N \subset \Lambda$.

Bonus problem (asteroids orbits). Implement a version of the asteroids game that illustrates the orbits of diagonal actions of $\mathbb{Z}$ on the torus $S^1 \times S^1$ whose factors are rotation actions.

- The player should be able to position the ship and to choose the two rotation angles.
- The bullets should then step by step visualise the corresponding orbit.

If you deploy it as a web application, then all course participants can enjoy it.

Submission before May 6, 2020, 8:00, via email to johannes.witzig@ur.de or through git. Solutions should be in PDF format and may be submitted in English or German.