Ergodic Theory of Groups: Week 5

Prof. Dr. C. Löh/J. Witzig May 13, 2020

Reading assignment (for the lecture on May 19).

- Read the rest of Chapter 1.5 *Orbit equivalence and measure equivalence.*
- This is the end of Chapter 1. If you have not done so already, this might be a good time to compile a summary of this chapter.
- Read Chapter 2.1.1 *Ergodicity.*
- Read Chapter 2.1.2 *Mixing actions.*

Reading assignment (for the lecture on May 20).

- (Optional) Recall group cohomology (in degree 0).
- Read Chapter 2.1.3 *Invariant bounded functions.*
- Recall basics on inner products and Hilbert spaces.
- Read Chapter 2.1.4 *Invariant L^2-functions* until Example 2.1.26.

Next week, we will study the so-called *ergodic theorems*, which are true classics of the field.

Exercises (for the session on May 22). The following exercises (which all are solvable with the material read/discussed in week 4) will be discussed.

Exercise 4.1 (Bernoulli shifts). Let Γ be a countable group, let $X := \{0,1\}^\Gamma$, and let μ be the product measure of the uniform distribution on $\{0,1\}$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. The standard Bernoulli shift $\Gamma \curvearrowright (X,\mu)$ is conjugate to the probability measure preserving action
$$\begin{aligned} \Gamma \times X &\longrightarrow X \\ (\gamma, x) &\longmapsto (\eta \mapsto x_{\gamma^{-1}\cdot\eta}) \end{aligned}$$
2. The standard Bernoulli shift $\Gamma \curvearrowright (X,\mu)$ is conjugate to the probability measure preserving action
$$\begin{aligned} \Gamma \times X &\longrightarrow X \\ (\gamma, x) &\longmapsto (\eta \mapsto x_{\gamma^{-1}\cdot\eta\cdot\gamma}) \end{aligned}$$

Exercise 4.2 (orbit relation).

1. Draw the orbit relation $\mathcal{R}_{R_{1/5}\colon \mathbb{Z}\curvearrowright S^1}$ in a suitable way.
2. Let $\Gamma \curvearrowright (X,\mu)$ be a standard probability action. Show that $\mathcal{R}_{\Gamma\curvearrowright(X,\mu)}$ is a measurable subset of $X \times X$.

Exercise 4.3 (induction). Let Γ be a countable group, let $\Lambda \subset \Gamma$ be a subgroup of finite index, and let $\Lambda \curvearrowright (Y,\nu)$ be a probability measure preserving action. Show the following:

1. If $\Lambda \curvearrowright (Y,\nu)$ is essentially free, then also $\Gamma \curvearrowright \operatorname{Ind}_\Lambda^\Gamma(\Lambda \curvearrowright (Y,\nu))$ is essentially free.
2. The action $\Gamma \curvearrowright \operatorname{Ind}_\Lambda^\Gamma(\Lambda \curvearrowright (Y,\nu))$ is stably orbit equivalent to $\Lambda \curvearrowright (Y,\nu)$.

Exercise 4.4 (OE cocycles). Let $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ be essentially free standard probability actions that are orbit equivalent through an almost everywhere defined map $f\colon (X,\mu) \longrightarrow (Y,\nu)$.

1. Show that the relation $f(\gamma\cdot x) = \alpha(\gamma,x)\cdot f(x)$ for $\gamma \in \Gamma$ and $x \in X$ gives an almost everywhere (well-)defined map $\alpha\colon \Gamma \times X \longrightarrow \Lambda$.
2. Show that $\alpha(\gamma_0\cdot\gamma_1, x) = \alpha(\gamma_0, \gamma_1\cdot x)\cdot\alpha(\gamma_1,x)$ holds for almost all $x \in X$ and all $\gamma_0,\gamma_1 \in \Gamma$.
3. Optional. How does α change if we choose a different map witnessing the orbit equivalence?

Bonus problem (universality of Bernoulli shifts).

1. Read the proof of the universality theorem of Abért and Weiss of Bernoulli shifts:

   M. Abért, B. Weiss. Bernoulli actions are weakly contained in any free action, *Ergodic Theory Dynam. Systems*, 33(2), pp. 323–333, 2013.
2. Sketch their proof.

Submission before May 20, 8:00, via email to johannes.witzig@ur.de or through git.