Ergodic Theory of Groups: Week 6

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Reading assignment (for the lecture on May 26).

- Read the rest of Chapter 2.1.4 *Invariant L^2-functions*.
- Read Chapter 2.2.1 *Averaging a single transformation*.
- Read Chapter 2.2.2 *The mean ergodic theorem* until Corollary 2.2.5.

Reading assignment (for the lecture on May 27).

- Recall the norms
  - $\|\cdot\|_\infty$ on $L^\infty(X,\mu)$,
  - $\|\cdot\|_1$ on $L^1(X,\mu)$.
  - $\|\cdot\|_2$ on $L^2(X,\mu)$.

  and relations between them.
- Read the rest of Chapter 2.2.2 *The mean ergodic theorem*.
- Read Chapter 2.2.3 *The pointwise ergodic theorem*.

Next week, we will consider applications of the ergodic theorems to decimal representations.

Implementation (preparation). We will soon-ish integrate the Isabelle aspect (the current forecast predicts this for the second week of June). Therefore, you should soon start with the following preparations:

- Install Isabelle 2019 or Isabelle 2020.

  https://www.cl.cam.ac.uk/research/hvg/Isabelle/installation.html
- (Optional) Adapt a suitable editor or IDE to cooperate with Isabelle. Personally, I prefer vscode over jedit (even though it also is far from perfect).
- Install the *Archive of Formal Proofs*.

  https://www.isa-afp.org/

Don't be bothered by the overall complexity of Isabelle! We will find a way through this step by step, once we start working with it ...

In case you don't have a machine that is able to run Isabelle, this means that you will not be able to try out the code we will write or to interactively develop code on your own. However, for the implementation exercises, you could still "program on paper"; unfortunately, for the programming language Isabelle this will be extremely challenging.

Exercises (for the session on May 29). The following exercises (which all are solvable with the material read/discussed in week 5) will be discussed.

Please turn over

Exercise 5.1 (ergodicity and products)**.** Let $\alpha\colon \Gamma \curvearrowright (X,\mu)$ and $\beta\colon \Gamma \curvearrowright (Y,\nu)$ be probability measure preserving actions and let $\alpha \times \beta\colon \Gamma \curvearrowright (X \times Y, \mu \otimes \nu)$ be the diagonal action. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If α and β are ergodic, then so is $\alpha \times \beta$.
2. If $\alpha \times \beta$ is ergodic, then α and β are ergodic.

Exercise 5.2 (ergodicity and induction)**.** Let Γ be a countable group, let $\Lambda \subset \Gamma$ be a finite index subgroup, and let $\beta\colon \Lambda \curvearrowright (Y,\nu)$ be a probability measure preserving action.

1. Show that $\operatorname{Ind}_\Lambda^\Gamma \beta$ is ergodic if β is ergodic.
2. What about the converse? Justify your answer.

Exercise 5.3 (mixing and products)**.** Let $\alpha\colon \Gamma \curvearrowright (X,\mu)$ and $\beta\colon \Gamma \curvearrowright (Y,\nu)$ be probability measure preserving actions and let $\alpha \times \beta\colon \Gamma \curvearrowright (X \times Y, \mu \otimes \nu)$ be the diagonal action. Show that if α and β are mixing, then $\alpha \times \beta$ is mixing.
Hints. Approximation!

Exercise 5.4 (mixing via L^2-functions)**.** Let Γ be a countable group. A unitary representation $\pi\colon \Gamma \longrightarrow U(H)$ on a Hilbert space H is *mixing* if for all $\xi, \eta \in H$, we have

$$\lim_{\Gamma \ni \gamma \to \infty} \bigl\langle \pi(\gamma)(\xi), \eta \bigr\rangle = 0.$$

Let $\Gamma \curvearrowright (X,\mu)$ be a probability measure preserving action. Show that the action $\Gamma \curvearrowright (X,\mu)$ is mixing if and only if the restriction of its Koopman representation to the orthogonal complement of the constant functions is mixing.
Hints. For the non-obvious implication: Approximation!

Bonus problem (shuffling cards)**.**

1. Look up the term *interval exchange transformation.*
2. How are they related to shuffling a deck of cards?
3. (When) Are interval exchange transformations mixing?
4. (When) Are interval exchange transformations ergodic?

Hints. Remember to cite the sources you use!