Ergodic Theory of Groups: Week 9

Prof. Dr. C. Löh/J. Witzig June 10, 2020

Reading assignment (for the lecture on June 16). In the following weeks, we will develop more of the theory of (measured) standard equivalence relations and their relations with dynamical systems.

- Read Chapter 3.1.3 *Measured equivalence relations.*
- Read Chapter 3.1.4 *Ergodicity.*
- (Optional) Recall basic notions/facts from *graph theory.*
- (Optional) Recall the notion of *Cayley graphs.*
- Read Chapter 3.2.1 *Graphings.*

Reading assignment (for the lecture on June 17).

- Read Chapter 3.2.2 *Cost of measured equivalence relations.*
- (Optional) Recall the (quantitative) *Nielsen-Schreier theorem.*
- Read Chapter 3.2.3 *Basic cost estimates* until Remark 3.2.16.

  It might be helpful to add more illustrations to the arguments.

Next week, we will establish more advanced computations of cost of measured equivalence relations and start with applications to groups.

Implementation (relations). Read the first Isabelle fragment: *Graphs of functions* (Appendix A.4); you might also want to interact with it:

http://www.mathematik.uni-r.de/loeh/teaching/erg_ss2020/1606/RelRel_Iso.thy

This fragment uses a tiny bit of the functionality of the Isabelle library Relation (which definitions/facts exactly?). Unfortunately, the interaction between functions and relations does not seem to be integrated into the Isabelle/HOL libraries (but only in Isabelle/ZF).

We might discuss this fragment in one of the lectures.

Exercises (for the session on June 19). The following exercises (which all are solvable with the material read/discussed in week 8) will be discussed.

Exercise 8.1 (standard equivalence relations)**.** Let $\mathcal{R}$ be a standard equivalence relation on a standard Borel space X. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

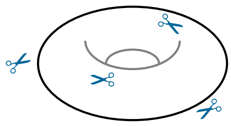
1. The set $\mathcal{R}$ is countable.
2. If $\mathcal{R}'$ is a standard equivalence relation on X, then so is $\mathcal{R} \cap \mathcal{R}'$.

Exercise 8.2 (ergodic decomposition of a torus)**.** We consider the self-map

$$\begin{aligned} S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ \big([x],[y]\big) &\longrightarrow \big([x+y],[y]\big) \end{aligned}$$

of the torus, which is a measurable isomorphism. Let $\mathbb{Z} \curvearrowright S^1 \times S^1$ be the corresponding action by measurable isomorphisms. Solve one of the following:

1. Give an ergodic decomposition of $\mathbb{Z} \curvearrowright (S^1 \times S^1, \mu)$, where μ is the product measure $\lambda \otimes \lambda$ of the Lebesgue measures on S^1.
2. Give an ergodic decomposition of $\mathbb{Z} \curvearrowright (S^1 \times S^1, \nu)$, where ν is the normalised counting measure on the set $\{([a/2020],[b/2020]) \mid a, b \in \mathbb{Z}\}$.



Exercise 8.3 (vanishing sequences of markers)**.** Solve one of the following:

1. Let $\mathcal{R}$ be an aperiodic standard equivalence relation on a countable set (with the discrete σ-algebra). Show that $\mathcal{R}$ admits a vanishing sequence of markers.
2. Let Γ be a residually finite infinite countable group. How can residual chains of Γ be turned into vanishing sequences of markers for $\mathcal{R}_{\Gamma \curvearrowright \widehat{\Gamma}}$?

Exercise 8.4 (implementation: orbits)**.** For now, we will stick to our implementation of equivalence relations in Isabelle. We consider the file

http://www.mathematik.uni-r.de/loeh/teaching/erg_ss2020/1006/Orbits_Exercise.thy

Complete this file, i.e.:

1. Give pen-and-paper completions of the definition of *orbit* and the missing parts of the proofs of *orbits_inclusion* and *orbits_equal*.
2. Implementation: Complete the definition of *orbit*.
3. Implementation: Complete the proof of *orbits_inclusion*.
4. Implemetnation: Complete the proof of *orbits_equal*.

Bonus problem (formal proofs)**.** Read the article

> L. Lamport. How to write a 21st century proof, *J. Fixed Point Theory Appl.*, 11(1), pp. 43–63, 2012

and answer the following questions:

1. Who is L. Lamport? I.e., what kind of research/tools is he known for?
2. What does he propose in this article? (A very short summary is enough.)

Submission before June 17, 8:00, via email to johannes.witzig@ur.de or through git.