Geometric Group Theory: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 1, April 26, 2022

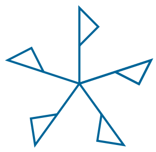
Quick check A (isomorphic groups?).

1. Are the additive groups $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ isomorphic?
2. Are the additive groups $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ isomorphic?

Quick check B (automorphism groups). Describe Galois groups and (if you know some algebraic topology) deck transformation groups as automorphism groups in suitable categories!

Quick check C (unfree groups). Use the universal property of free groups to show that the groups $\mathbb{Z}/2022$ and $\mathbb{Z}^{2022}$ are *not* free.

Quick check D (isometry groups). Determine the isometry group of the following subset of $\mathbb{R}^2$ with respect to the Euclidean metric. How could one turn this argument into a rigorous proof? Generalise!



Quick check E (groups). What is your favourite example of a group? Do you know "exotic" examples of groups?

Exercise 1 (symmetric groups; 4 credits). Let X be an infinite set. Is the symmetric group S_X finitely generated? Justify your answer!

Exercise 2 (rank of free groups; 4 credits). Let S be a set, let F be the free group generated by S. Prove that if $T \subset F$ is a generating set of F, then $|T| \geq |S|$.
Hints. If you want, you may restrict to the case that S is finite.

Exercise 3 (the normal subgroup trick; 8 credits). Let G be a group.

1. Let $H, K \subset G$ be subgroups of finite index. Show that $H \cap K$ also has finite index in G.
2. Let $H \subset G$ be a subgroup and let $S \subset G$ be a set of representatives of $\{g \cdot H \mid g \in G\}$. Show that
$$\bigcap_{g \in G} g \cdot H \cdot g^{-1} = \bigcap_{g \in S} g \cdot H \cdot g^{-1}.$$
3. Let $H \subset G$ be a subgroup of finite index. Show that there exists a normal subgroup $N \subset G$ of finite index with $N \subset H$.

Bonus problem (mapping class groups; 4 credits). Look up the term *mapping class group* (e.g., of manifolds, topological spaces); give a reference for the definition you found. Which formal similarity is there between this definition and the definition of outer automorphisms of groups?

Submission before May 3, 2022, 8:30, via GRIPS (in English or German)
The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on May 2, 2022.