Geometric Group Theory: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 10, June 28, 2022

Quick check A (sub-multiplicativity of growth functions)**.** Let G be a finitely generated group and let $S \subset G$ be a finite generating set. Show that then $\beta_{G,S}(r+r') \leq \beta_{G,S}(r) \cdot \beta_{G,S}(r')$ holds for all $r, r' \in \mathbb{N}$.

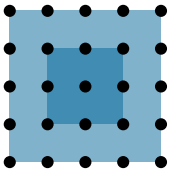
Quick check B (balls in free groups)**.** Pick a (non-free!) finite generating set of a free group of rank 2 and illustrate the balls around the neutral element.

Quick check C (balls in $\mathbb{Z}^2$)**.**

1. Is there a finite generating set $S \subset \mathbb{Z}^2$ with $\beta_{\mathbb{Z}^2,S}(42) = 2022$?

   Hints. Parity!

2. Is there a finite generating set $S \subset \mathbb{Z}^2$ that satisfies $B_r^{\mathbb{Z}^2,S}(e) = \{-r, \dots, r\}^2$ for all $r \in \mathbb{N}$?



Exercise 1 (growth of $\mathbb{Z}^n$; 4 credits)**.** Let $n \in \mathbb{N}$. Show that $\mathbb{Z}^n$ has the growth type of $(x \mapsto x^n)$. Illustrate your arguments!

Exercise 2 (growth functions of infinite groups; 4 credits)**.** Let G be a finitely generated infinite group and let $S \subset G$ be a finite generating set. Show that $\beta_{G,S}(r) \geq r$ for all $r \in \mathbb{N}$. Illustrate your arguments!

Exercise 3 (geometric properties?; 8 credits)**.** Which of the following properties of finitely generated groups are geometric? Justify your answers!

1. containing a generating set with at most 2022 elements;
2. being isomorphic to a subgroup of $\mathbb{Z}^{2022}$;
3. being isomorphic to a subgroup of $\mathrm{SL}(2,\mathbb{Z})$;
4. being isomorphic to a free product of two non-trivial groups;

Bonus problem (growth type of the Heisenberg group; 8 credits)**.** Let $H \subset \mathrm{SL}(3,\mathbb{Z})$ be the Heisenberg group. Show that H has the growth type of $(x \mapsto x^4)$:

0. Show that $H \cong \langle x, y, z \mid [x,z], [y,z], [x,y] = z \rangle$.

In the following, we write $S := \{x, y, z\}$ and view S as a subset of H. Let $m, n, k \in \mathbb{Z}$.

1. Show that $d_S(x^m \cdot y^n \cdot z^k, e) \leq |m| + |n| + 6 \cdot \sqrt{|k|}$.
2. Show that $|m| + |n| \leq d_S(x^m \cdot y^n \cdot z^k, e)$ and $|k| \leq d_S(x^m \cdot y^n \cdot z^k, e)^2$.
3. Show that $1/2 \cdot (|m| + |n| + \sqrt{|k|}) \leq d_S(x^m \cdot y^n \cdot z^k, e)$.
4. Conclude that the growth function $\beta_{H,S}$ is quasi-equivalent to a polynomial of degree 4.

Submission before July 5, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on July 4, 2022.