Geometric Group Theory: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 11, July 5, 2022

Quick check A (polynomial growth?)**.** Which of the following groups have polynomial growth?

$$\mathbb{Z}^{2022}, \quad \mathbb{Z}^{2022} \times \mathbb{Z}/2, \quad \mathbb{Z}^{2022} * \mathbb{Z}/2, \quad \mathbb{Z}/2 * \mathbb{Z}/2, \quad \mathrm{SL}(2,\mathbb{Z})$$

Quick check B (non-polynomial growth)**.** Let G be a finitely generated infinite group that is quasi-isometric to $G \times G$. Show that G does *not* have polynomial growth.

Quick check C (exponential growth?)**.** Let G be a group that admits an epimorphism to a free group of rank 2. Does G have exponential growth?

Exercise 1 (exponential growth?!; 4 credits)**.** Does the group

$$\bigl\langle x_1, \dots, x_{2022} \bigm| [x_1, x_2] \cdot [x_3, x_4] \cdot \dots \cdot [x_{2021}, x_{2022}] \bigr\rangle$$

have exponential growth? Justify your answer!

Exercise 2 (solvable groups of exponential growth; 8 credits)**.** Let $n \in \mathbb{N}_{>1}$ and let $A \in \mathrm{GL}(n,\mathbb{Z})$ be a matrix that over $\mathbb{C}$ has an eigenvalue $\lambda \in \mathbb{C}$ with $|\lambda| \geq 2$. We consider the associated semi-direct product $G_A = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ with respect to the homomorphism $\mathbb{Z} \longrightarrow \operatorname{Aut} \mathbb{Z}^n$ given by the action of the powers of A on $\mathbb{Z}^n$ by matrix multiplication.

1. Show that there exists an $x \in \mathbb{Z}^n$ with: If $k \in \mathbb{N}$, then the 2^{k+1} elements $\sum_{j=0}^{k} \varepsilon_j \cdot A^j \cdot x$ of $\mathbb{Z}^n$ with $\varepsilon_0, \dots, \varepsilon_k \in \{0,1\}$ are all different.

   Hints. To simplify matters, you may restrict to the case that the "largest" eigenvalue of A is real.

2. Conclude that G_A has exponential growth.

Exercise 3 (finite generation and projections on $\mathbb{Z}$; 4 credits)**.** Let G be a finitely generated group with subexponential growth that admits a surjective homomorphism $\pi \colon G \longrightarrow \mathbb{Z}$. Show that then the kernel of π is also finitely generated.
Hints. Choose an element $g \in G$ with $\pi(g) = 1$. Show that there is a finite subset $S \subset \ker \pi$ such that $\{g\} \cup S$ generates G. For $s \in S$ and $n \in \mathbb{N}$, let $g_{n,s} := g^n \cdot s \cdot g^{-n} \in \ker \pi$. Prove that there is an $N \in \mathbb{N}$ such that $\ker \pi$ is generated by the (finite!) set $\{g_{n,s} \mid s \in S, n \in \{-N, \dots, N\}\}$ by considering elements of the form "$g_{n_0,s^{\varepsilon_0}} \cdot \dots \cdot g_{n_k,s^{\varepsilon_k}}$".

Bonus problem (die Hilbertschen Probleme; 4 credits)**.** On August 8, 1900, David Hilbert gave his famous speech *Mathematische Probleme* (Mathematical Problems) at the International Congress of Mathematicians in Paris. These problems are now known as *Hilbert's problems*. Take a random number n between 1 and 23. Describe Hilbert's n-th problem and the status of its solution. Do not forget to cite sources properly!

Submission before July 12, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on July 11, 2022.