

Geometric Group Theory: Exercises

Prof. Dr. C. Löh/M. Uschold

Sheet 12, July 12, 2022

Quick check A (exponential growth rate). Let $H_3 \subset \mathrm{SL}(3, \mathbb{Z}/3)$ be the image of the Heisenberg group $H \subset \mathrm{SL}(3, \mathbb{Z})$ under the reduction $\mathbb{Z} \rightarrow \mathbb{Z}/3$ and let $S \subset H_3$ be a finite generating set. What is the exponential growth rate of H_3 with respect to S ?

Quick check B (exponential growth rate in free groups). Let F be a free group of rank 2. Show that there exist finite generating sets $S, T \subset F$ with $\varrho_{F,S} \neq \varrho_{F,T}$.
Hints. There is a solution that requires no calculations.

Quick check C (2022-hyperbolicity). Is every hyperbolic metric space also a 2022-hyperbolic metric space?

Exercise 1 (groups of uniform exponential growth; 4 credits). Let G be a finitely generated group of uniform exponential growth. Does then also $G \times G$ have uniform exponential growth? Justify your answer!

Exercise 2 (groups and roots; 4 credits). Let $\alpha, \beta \in \mathbb{C}$. We consider

$$A(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \quad B(\beta) := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

Prove that if α is *not* a root of unity, then the group $\langle A(\alpha), B(1) \rangle_{\mathrm{GL}(2, \mathbb{C})}$ is *not* virtually nilpotent.

Hints. Calculate $[A(\alpha^n), B(\beta)]$. Why does this help?

Exercise 3 (a weird quasi-geodesic ray in the Euclidean plane; 8 credits).

1. Show that the following map is a quasi-isometric embedding with respect to the standard metrics on \mathbb{R} and \mathbb{R}^2 , respectively:

$$\begin{aligned} \mathbb{R}_{\geq 0} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto t \cdot (\sin(\ln(1+t)), \cos(\ln(1+t))) \end{aligned}$$

2. Conclude that the stability theorem for quasi-geodesics does not hold in the Euclidean space \mathbb{R}^2 .



Bonus problem (Fekete's lemma; 4 credits). Translate the proof of Fekete's lemma given in the Lean mathlib library `analysis.subadditive` into a pen-and-paper version (all definitions, statements, and proofs). Moreover, explain how Fekete's lemma can be used to show that the exponential growth rate is well-defined.

Submission before July 19, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on July 18, 2022.

This is the last regular exercise sheet. Subsequent sheets will give bonus credits.