Geometric Group Theory: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 13, July 19, 2022

Quick check A (summary)**.** What is your favourite group? For this group,

1. determine the growth type;
2. decide whether it is hyperbolic or not;
3. decide whether it is quasi-isometric to $\mathbb{R}^3$ or not;
4. determine whether it is finitely presented; if so, give a finite presentation.

Quick check B (quadrilaterals)**.** In a quadrilaterals, which triangles appear naturally? How could this help in hyperbolic spaces?

Exercise 1 (local geodesics in general spaces; 4 credits)**.** Let X be a metric space and let $\gamma\colon [0,2022] \longrightarrow X$ be a 1-local geodesic. Is γ then necessarily a geodesic? Justify your answer!

Exercise 2 (geodesics in hyperbolic spaces starting at the same point; 4 credits)**.** Let $\delta, D \in \mathbb{R}_{\geq 0}$, let (X,d) be a δ-hyperbolic space, and let $\gamma\colon [0,L] \longrightarrow X$, $\gamma'\colon [0,L'] \longrightarrow X$ be geodesics in X with $\gamma'(0) = \gamma(0)$ and $d(\gamma'(L'), \gamma(L)) \leq D$. Show that γ and γ' are uniformly $(2 \cdot \delta + D)$-close, i.e.,

$$\forall_{t \in [0,\min(L,L')]} \quad d\big(\gamma(t), \gamma'(t)\big) \leq 2 \cdot \delta + D \quad \text{and} \quad |L - L'| \leq D.$$

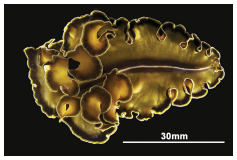
Illustrate your proof by suitable pictures!
Hints. Distinguish the different cases arising from δ-slimness!

Exercise 3 (local geodesics in hyperbolic spaces; 8 credits)**.** Let X be a δ-hyperbolic space and let $c \in \mathbb{R}_{>8\delta}$. Let $\gamma\colon [0,L] \longrightarrow X$ be a c-local geodesic and let $\gamma'\colon [0,L'] \longrightarrow X$ be a geodesic with $\gamma'(0) = \gamma(0)$ and $\gamma'(L') = \gamma(L)$. Prove that

$$\operatorname{im}\gamma \subset B_{2\cdot\delta}(\operatorname{im}\gamma').$$

Illustrate your proof by suitable pictures!
Hints. Consider a point in $\operatorname{im}\gamma$ that has maximal distance from $\operatorname{im}\gamma'$ and then look at a suitable geodesic quadrilateral that connects $\operatorname{im}\gamma$ and $\operatorname{im}\gamma'$ and that contains this point on one of its sides. Exclude the "weird" cases by computation.

Bonus problem (real hyperbolicity; 4 credits)**.** Crochet/knit a hyperbolic sphere or annulus! For submission: Take pictures of at least two intermediate stages and from at least two different perspectives of the completed model.
Hints. https://pi.math.cornell.edu/~dwh/papers/crochet/crochet.html
How ironic that marine flatworms are called *flat*worms!



Pseudobiceros flowersi
https://commons.wikimedia.org/wiki/File:Pseudobiceros_flowersi_%2810.11646-zootaxa.4019.1.14%29_Figure_7_%28cropped%29.png (CC 3.0)

Submission before July 26, 2022, 8:30, via GRIPS (in English or German)
The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on July 25, 2022.
All credits on this sheet count as bonus credits.