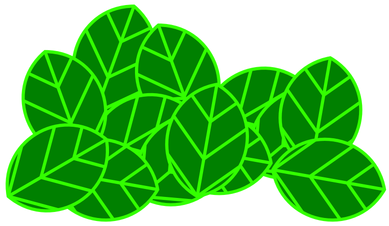
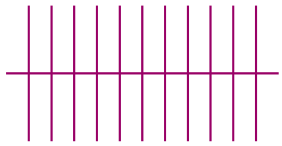
Geometric Group Theory: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 14, July 26, 2022

Commander Blorx was summoned to the Digit Council for interrogation. When asked for the result of $9 \cdot 6$, Blorx blacked out, forgot that the actual answer is 42, then decided to compute the result on the spot, but confused the German and English ordering when reading numbers and replied 45. The highest member of the Digit Council, Emperor Neun, thus was not amused and sentenced Blorx to solve the word problem in the infamous jungle on the planet Ji-Ji-Tee. Help Blorx! Beware: Multiple answers might be correct ...

Problem 1 (the centipede). Hidden under a small rock Blorx finds a centipede. According to the latest research, centipedes on Ji-Ji-Tee do not have 100 legs but are graphs of the form $\bigl(V, \{\{v,w\} \mid v,w \in V,\ \|v-w\|_1 = 1\}\bigr)$ on the vertex set $V := ((2 \cdot \mathbb{Z}) \times \mathbb{Z}) \cup (\mathbb{Z} \times \{0\}) \subset \mathbb{R}^2$.



Which groups are quasi-isometric to the centipede?

$\mathbb{Z}$	OR
$\mathbb{Z}^2$	YOU
F_2	TH
There is no such group	FOR

Problem 2 (eek, spiders!). Blorx journeys onward. Suddenly he realises that the little black sticks on the ground are moving and in fact are spider legs. Lots of them. The spiders are interlocked in an 8-regular tree. Which groups could have left such a Cayley mess?

$\mathrm{SL}(3,\mathbb{Z})$	GR
F_4	THR
$\mathbb{Z}^5$	IN
There is no such group	FRE

Problem 3 (the sloth). After the previous encounters Blorx decides that it might be safer to travel further up through the trees. However, there the way is blocked by a particularly lazy sloth, which only travels a generator a day. In the tree

$$\langle \mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{H}, \mathsf{N}, \mathsf{R} \mid \mathsf{NA}^2 = \mathsf{A}^3\mathsf{N}^2, \mathsf{AN} = \mathsf{A}^3\mathsf{N}^2\mathsf{A}^{-2}, \mathsf{RAN} = \mathsf{HAH}^{-1}\mathsf{A} \rangle,$$

will the sloth reach the BRANCH within three days (and thus let Blorx pass before he starves)?

Yes	HEC
No	UP

Problem 4 (the monkey puzzle tree). The next trees that Blorx reaches are much more lively – being conquered by a monkey population. Can the group

$$\langle \mathsf{A}, \mathsf{C}, \mathsf{I}, \mathsf{N}, \mathsf{R}, \mathsf{U} \mid \mathsf{ARAUCARIA} = \mathsf{ARAUCANA} \rangle$$

of monkeys act freely on a non-empty tree?

Yes	UPO
No	TR

Problem 5 (more trees!). Obviously, the monkeys need more space, i.e., more trees. Which of the following seeds are not suited for this purpose because they grow only polynomially?

$\mathbb{Z}/3 \times H$	EEN
$H \times H$	OTT
$H * H$	REE
$\mathrm{SL}(3, \mathbb{Z})$	INN
There is no such group in this list	ATT

Problem 6 (the mantis lord). In any case, travelling through the trees didn't turn out as relaxing as Blorx hoped. He thus jumps into the water. There, he barely escapes the hyperbolicity-fuelled (seriously!) kick of a mantis shrimp. Which of the following groups are hyperbolic and could thus be used by Blorx for defense?

$\mathbb{Z}/2 * \mathbb{Z}/2$	ICG
$\mathbb{Z}^2 * \mathbb{Z}^2$	IS
$H * \mathrm{SL}(2, \mathbb{Z})$	RI
$F_2 * F_2$	RO
$F_2 * \mathrm{SL}(3, \mathbb{Z})$	NO
There is no such group in this list	EK

Problem 7 (action!). The continuing problems on land, trees, and water cause a sense of discomfort and urgency in Blorx. He starts running. Quasi-geodesically. In which groups does *not* every element g of infinite order induce a quasi-geodesic line through $\mathbb{Z} \ni n \longmapsto g^n$?

$\mathbb{Z}/3$	LEM
$\mathbb{Z}^2$	AC
$H * \mathbb{Z}^2$	YCL
F_2	SO
There is no such group in this list	MA

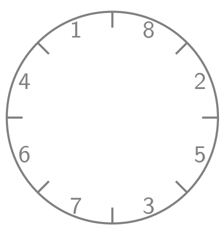
Problem 8 (the temple). Completely exhausted, Blorx reaches a temple. The temple bears a mysterious inscription:

$$\langle \mathsf{I}, \mathsf{F}, \mathsf{O}, \mathsf{R}, \mathsf{T}, \mathsf{U}, \mathsf{Y} \mid \mathsf{FOUR} = \mathsf{FIVE}, \mathsf{FIFTY} = \mathsf{FORTY} \rangle$$

Clearly, like every other jungle temple, this temple must have been built by aliens that are far more enlightened than Emperor Neun and the temple serves only as camouflage for an advanced rocket launchpad! To unlock the gate and hence reach the launchpad, Blorx needs to decide whether there is a group homomorphism from the inscription to $\mathbb{Z}$ that maps $\mathsf{FIFTYFOUR}$ to 45. Does such a homomorphism exist?

Yes	DER
No	DIE

Solution (clock-wise):



No submission! Enjoy the summer break!