Geometric Group Theory: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 2, May 3, 2022

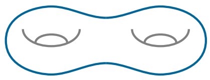
Hints. Do not resort to dodgy arguments with generators and relations, but use universal properties whenever appropriate!

Quick check A (a presentation of $\mathbb{Z}^2$)**.** Show that $\langle x, y \mid xyx^{-1}y^{-1} \rangle \cong \mathbb{Z}^2$.

Quick check B (trivial groups?)**.** Which of the following groups are trivial?

1. $\langle x, y \mid xyx = yxy \rangle$
2. $\langle x, y \mid yx^{2022}y = x^{2021}, x^2y = x \rangle$
3. $\langle x, y \mid xy^{2022}x = yx^{2021}, x^{2022} = y^{2021} \rangle$
4. $\bigl\langle a_1, a_2, b_1, b_2 \bigm| [a_1, b_1][a_2, b_2] \bigr\rangle$

Quick check C (fundamental groups)**.** If you know fundamental groups: Which of the presentations above are related to the following picture?



Exercise 1 (the infinite dihedral group; 4 credits)**.** The *infinite dihedral group* is defined as

$$D_\infty := \langle s, t \mid t^2, tst^{-1} = s^{-1} \rangle.$$

Show that $D_\infty \cong \operatorname{Isom}(\mathbb{Z}, d)$, where d is the metric on $\mathbb{Z}$ induced by the absolute value on $\mathbb{R}$.

Exercise 2 (an alternative presentation of the infinite dihedral group; 4 credits)**.** Show that

$$D_\infty \cong \langle x, y \mid x^2, y^2 \rangle$$

Hints. Try first to find geometric candidates for such generators!

Exercise 3 (finite normal generation of kernels; 8 credits)**.** Let $\varphi \colon G \longrightarrow H$ be a surjective group homomorphism, where G is finitely generated and H is finitely presented. Show that there exists a finite set $N \subset G$ with

$$\ker \varphi = \langle N \rangle^{\triangleleft}_G$$

Hints. Start with a finite generating set of G and then find a finite presentation of H that is related to this generating set. Structure your proof into steps.

Bonus problem (a finitely generated group without finite presentation; 4 credits)**.** We consider the group

$$G := \bigl\langle s, t \bigm| \{[s, t^n s t^{-n}] \mid n \in \mathbb{N}_{>0}\} \bigr\rangle.$$

Show that G is *not* finitely presentable.

Hints. For $N \in \mathbb{N}_{>0}$, let $G_N := \langle s, t \mid \{[s, t^n s t^{-n}] \mid n \in \{1, \dots, N\}\} \rangle$. Show that the homomorphism $\pi_N \colon G_N \longrightarrow G_{N+1}$ induced by the identity on $\{s, t\}$ is surjective but *not* injective. Homomorphisms to S_{2N+3} might help.

Submission before May 10, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on May 9, 2022.