# Geometric Group Theory: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 3, May 10, 2022

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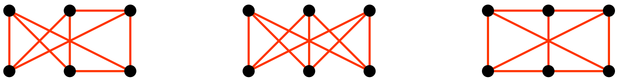
**Quick check A** (free products)**.** Show that $\mathbb{Z} * \mathbb{Z}$ is free of rank 2.

**Quick check B** (HNN-extensions)**.** Establish an appropriate universal property for HNN-extensions!

**Quick check C** (group sudoku)**.** Play a round of *group sudoku* by Raphael Appenzeller: https://n.ethz.ch/~apraphae/sudokuformathematicians.html

**Quick check D** (Isomorphic graphs?)**.** Which of the following graphs are isomorphic?



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**Exercise 1** (pushout groups; 4 credits)**.** Let

$$\begin{tikzcd} A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & G_1 \arrow[d, "\beta_1"] \\ G_2 \arrow[r, "\beta_2"'] & G \end{tikzcd}$$

be a pushout diagram of groups. Are then $\beta_1$ and $\beta_2$ necessarily injective? Justify your answer by a proof or counterexample!

**Exercise 2** (ascending HNN-extensions; 4 credits)**.** Let $G$ be a group and let $\vartheta \in \operatorname{Aut}(G)$. Show that $G*_\vartheta$ is isomorphic to a semi-direct product of $\mathbb{Z}$ with kernel $G$.

**Exercise 3** (spanning trees; 8 credits)**.** Use Zorn's lemma to show that every connected non-empty graph contains a spanning tree.
*Hints.* A *spanning tree* is a subgraph that is a tree and contains all vertices.

**Bonus problem** (social networks; 4 credits)**.**

1. How can graphs be used to model social networks?
2. Give an example of how personal information could be inferred from knowledge of such a graph.

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Submission before May 17, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on May 13(!), 2022.