

Geometric Group Theory: Exercises

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Sheet 4, May 17, 2022

Quick check A (Cayley graphs). Pick your favourite group and your favourite generating set. Sketch the associated Cayley graph!

If your favourite group happens to be $\mathbb{Z}/3$, you should give D_{2022} (and a proper generating set) a try!

Quick check B (powers in free groups). Let F be a free group of rank 2.

1. Which elements $g \in F$ satisfy $g^2 = e$?
2. Are there elements $g, h \in F$ with $g^{2022} = h^{2023}$?

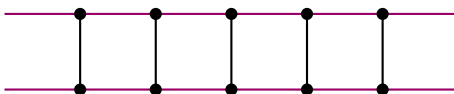
Quick check C (actions on trees).

1. Is every action of a free group on a tree free?
 2. Can free groups act freely on graphs that are not trees?
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Exercise 1 (Cayley graph? 4 credits). Does there exist a group that has a Cayley graph with exactly 2022 vertices and exactly 2023 edges? Justify your answer!

Exercise 2 (isomorphic Cayley graphs; 4 credits). Show that there exist finite generating sets S of $\mathbb{Z} \times \mathbb{Z}/2$ and T of D_∞ such that the graphs $\text{Cay}(\mathbb{Z} \times \mathbb{Z}/2, S)$ and $\text{Cay}(D_\infty, T)$ are isomorphic.

Hints. Do not only draw pictures, but give an actual proof!



Exercise 3 (actions of finite groups on trees; 8 credits). Prove (without using the characterisation of free groups in terms of free actions on trees) that every action of a finite group G on a non-empty tree T has a global fixed point (i.e., a vertex or an edge on which all group elements act trivially). Proceed in the following steps:

1. Why/How can one restrict to the case that T is finite?
2. A vertex of a tree of degree 1 is called a *leaf*. Show that removing the leaves of T produces a tree T' and that the G -action on T restricts to an action on T' .
3. Use the previous step to shrink the original tree and conclude.

Bonus problem (Women of Mathematics throughout Europe; 4 credits). Pick four of the posters of the exhibition *Women of Mathematics throughout Europe* in the math building and for each of them list the name, the current affiliation (might not be the same as on the poster), the research field, and the title and full reference of one published paper of the corresponding researcher (the database <https://mathscinet.ams.org> might help).

Submission before May 24, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on May 23, 2022.