

# Geometric Group Theory: Exercises

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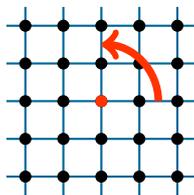
Sheet 5, May 24, 2022

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**Quick check A** (faithful vs. free actions). Let  $G$  be a group, let  $C$  be a category, and let  $X$  be an object of  $C$ . A group action  $\varrho: G \rightarrow \text{Aut}_C(X)$  of  $G$  on  $X$  in  $C$  is *faithful* if  $\varrho$  is a monomorphism.

1. Is every free group action on a non-empty set faithful?
2. Is every faithful group action on a non-empty set free?

**Quick check B** (spanning trees). The group  $\mathbb{Z}/4$  acts on  $\text{Cay}(\mathbb{Z}^2, \{(1, 0), (0, 1)\})$  by rotation about  $\pi/2$  around 0. Sketch a spanning tree for this group action! Is the spanning tree of your neighbour isomorphic to yours?



**Quick check C** (rank gradient). Let  $F$  be a free group of finite rank. Compute the following number:

$$\inf \left\{ \frac{\text{rk } H}{[F : H]} \mid H \subset F \text{ is a subgroup of finite index} \right\}$$

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**Exercise 1** (counting in finite trees; 4 credits). Let  $(V, E)$  be a finite non-empty tree. Show that

$$|E| = |V| - 1.$$

**Exercise 2** (subgroups of large rank; 4 credits). Let  $F$  be a finitely generated free group of rank at least 2. Prove: For each  $n \in \mathbb{N}$ , there exists a finitely generated free subgroup  $G$  of  $F$  of rank at least  $n$ .

**Exercise 3** (characterisation of finite cyclic groups; 8 credits). Find a class  $C$  of graphs with the following property: A group is finite cyclic (i.e., generated by an element of finite order) if and only if it admits a free action on some graph  $C$ . Prove your claim!

*Hints.* What are “the” special Cayley graphs of finite cyclic groups?

**Bonus problem** (the Hanna Neumann conjecture; 4 credits).

1. What is the statement of the Hanna Neumann conjecture?
  2. Where was the Hanna Neumann conjecture originally formulated? Give the reference!
  3. What is the statement of the strengthened Hanna Neumann conjecture?
  4. Give two references that contain (different) proofs of the strengthened Hanna Neumann conjecture!
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Submission before May 31, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on May 30, 2022.