Geometric Group Theory: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 6, May 31, 2022

Quick check A (ping-pong?). Let G be a group generated by elements a and b. Suppose there is a G-action on a set X such that there are non-empty subsets A, $B \subset X$ with B not contained in A and such that

$$a \cdot B \subset A \quad \text{and} \quad b \cdot A \subset B.$$

Is then G free of rank 2 ? Justify your answer!

Quick check B (unsolvability of free groups). Show that the free group of rank 2 is *not* solvable.

Quick check C (maps close to quasi-isometric embeddings). Show that every map at finite distance to a quasi-isometric embedding is a quasi-isometric embedding.

Exercise 1 (eigen-ping-pong; 4 credits). Let $\lambda \in \mathbb{C}$. We consider the matrices

$$a := \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \qquad c := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad b := c \cdot a \cdot c^{-1}$$

in $\mathrm{GL}(2,\mathbb{C})$. Use the action of $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{C}^2$ by matrix multiplication and the sets

$$B := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \frac{|x|}{|y|} \in (1-\varepsilon, 1+\varepsilon) \right\} \quad \text{and} \quad A := c \cdot B.$$

to show that $\langle \{a, b\} \rangle_{\mathrm{GL}(2,\mathbb{C})}$ is free of rank 2 provided $|\lambda|$ is "big" and ε is "small". Make "big" and "small" precise and illustrate the situation by suitable pictures (over $\mathbb{R}$).

Exercise 2 (free groups are residually finite; 8 credits). A group G is *residually finite* if for every $g \in G \setminus \{e\}$, there exists a finite group H and a group homomorphism $\varphi \colon G \longrightarrow H$ with $\varphi(g) \neq e$.

1. Show that the group $\mathrm{SL}(2,\mathbb{Z})$ is residually finite.
2. Conclude that free groups of rank 2 are residually finite.
3. Conclude that all free groups are residually finite.

Exercise 3 (quasi-isometry? 4 credits). Are the spaces $\mathbb{Z}$ and $\{n^3 \mid n \in \mathbb{Z}\}$ (with the standard metric on $\mathbb{R}$) quasi-isometric? Justify your answer!

Bonus problem (free rotation groups; 4 credits). Show that the special orthogonal group $\mathrm{SO}(3)$ contains a free subgroup of rank 2.

Hints. Consider the matrices

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

and divisibility by 5 in $\mathbb{Z}^3 \subset \mathbb{R}^3$.

Submission before June 8(!), 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on Thu, June 9(!), 16:15–17:45, H32.