Geometric Group Theory: Exercises

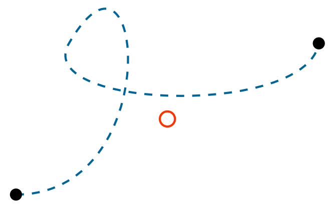
Prof. Dr. C. Löh/M. Uschold Sheet 8, June 14, 2022

Quick check A (quasi-dense subgroups). Let G be a finitely generated group with finite generating set $S \subset G$ and let $H \subset G$ be a subgroup that is quasi-dense in G with respect to d_S. Show that H has finite index in G.

Quick check B (the maximum metric). Show that $(\mathbb{R}^2, d_\infty)$ is a geodesic metric space.

Quick check C (the punctured plane: quasi-geodesics). Let $\varepsilon \in \mathbb{R}_{>0}$. Show that $\mathbb{R}^2 \setminus \{0\}$ is $(1, \varepsilon)$-quasi-geodesic with respect to the standard metric.

Exercise 1 (the punctured plane: geodesics; 4 credits). Prove(!) that $\mathbb{R}^2 \setminus \{0\}$ is *not* geodesic with respect to the standard metric.



Exercise 2 (Švarc–Milnor lemma; 8 credits). For each of the following group actions name one of the conditions of the Švarc–Milnor lemma that is satisfied, and one that is not (and prove your claims).

1. The action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ by matrix multiplication.
2. The action of $\mathbb{Z}$ on $X := \{(r^3, s) \mid r, s \in \mathbb{Z}\}$ (with the standard metric on $\mathbb{R}^2$) given by
$$\begin{aligned} \mathbb{Z} \times X &\longrightarrow X \\ \bigl(n, (r^3, s)\bigr) &\longmapsto (r^3, s + n). \end{aligned}$$

Exercise 3 (the Heisenberg group; 4 credits). Let $H_\mathbb{R}$ be the real Heisenberg group and let $H \subset H_\mathbb{R}$ be the Heisenberg group:
$$H_\mathbb{R} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \quad \text{and} \quad H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\}.$$
We equip $H_\mathbb{R}$ with the topology given by convergence of all matrix coefficients. Show that $H \subset H_\mathbb{R}$ is discrete with respect to this topology and that the left translation action of H on $H_\mathbb{R}$ is cocompact (i.e., that the quotient $H \setminus H_\mathbb{R}$ is compact with respect to the quotient topology).

Bonus problem (Švarc–Milnor lemma via quasi-isometric actions; 4 credits). Formulate and prove a truly quasi-geometric version of the Švarc–Milnor lemma, i.e., a version of the Švarc–Milnor lemma where the given group action is an action by quasi-isometries instead of isometries.

Submission before June 21, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on June 20, 2022.