# Seminar WS 2024/25

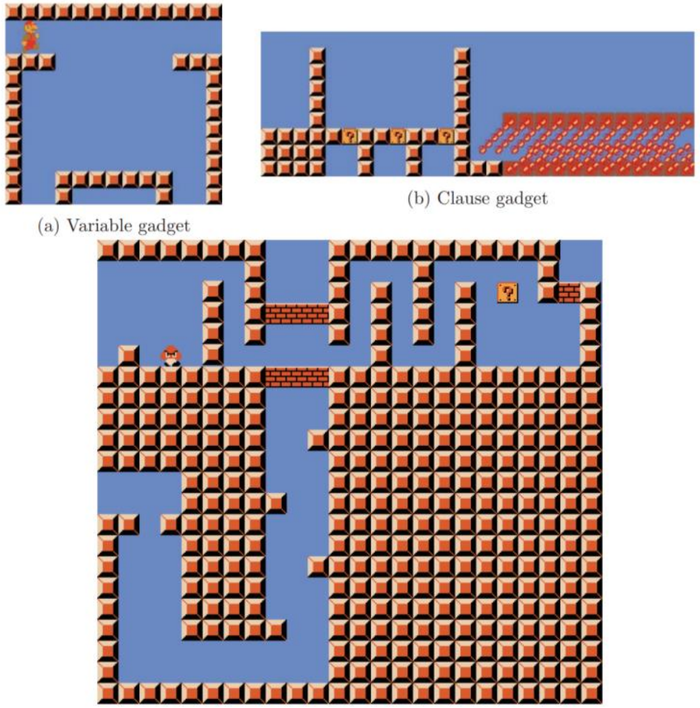
# Graphs, Groups, Topology, and Computational Complexity

Clara Löh
Radu Curticapean

P

NP

P



(a) Variable gadget

(b) Clause gadget

(c) Crossover gadget

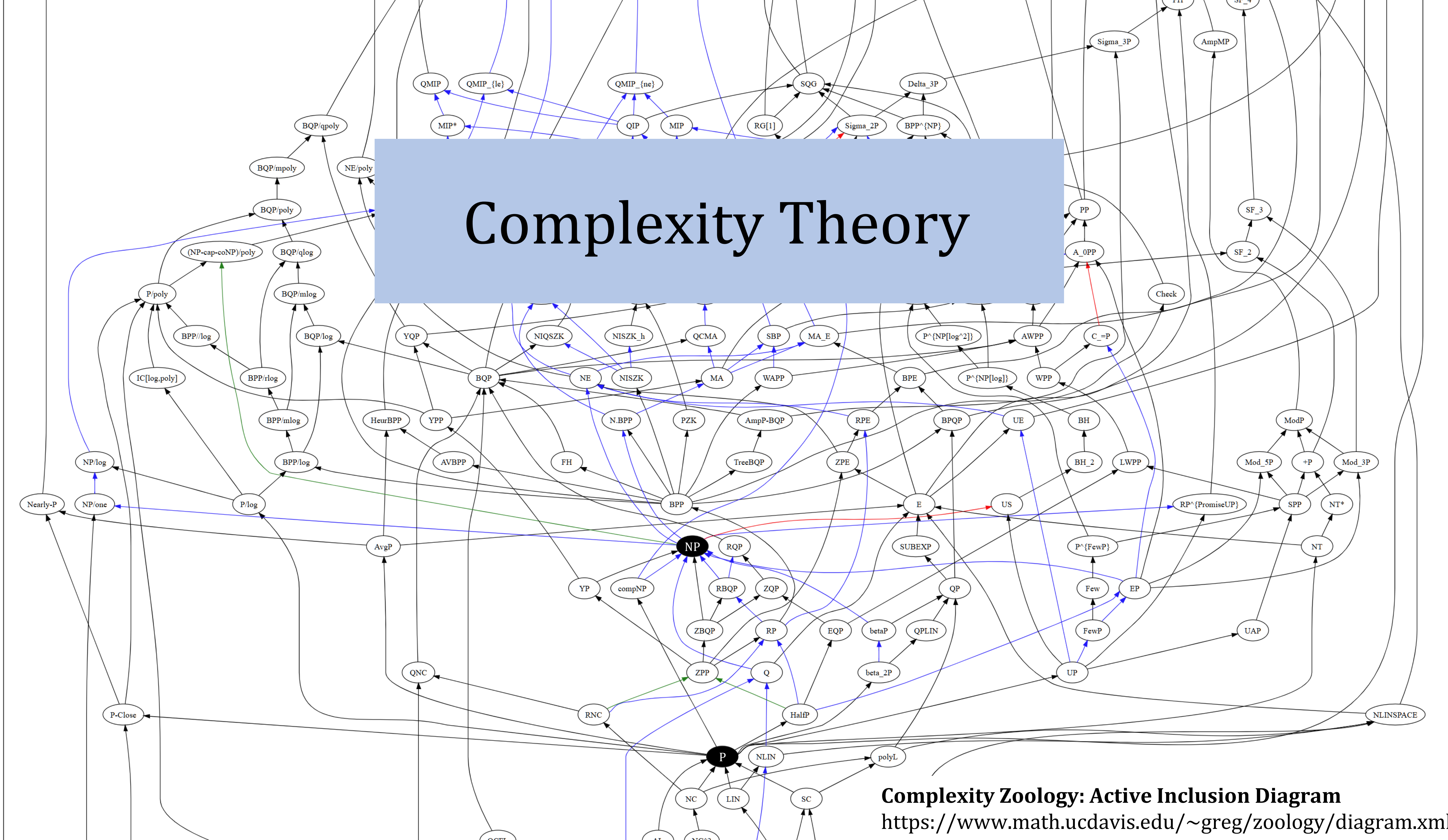
Figure 1: Gadgets for Super Mario Brothers.

NP

**Classic Nintendo Games are (Computationally) Hard**

Greg Aloupis, Erik D. Demaine, Alan Guo, Giovanni Viglietta

# Complexity Theory




**Complexity Zoology: Active Inclusion Diagram**
https://www.math.ucdavis.edu/~greg/zoology/diagram.xml

upper bounds on resources
(efficient algorithms)

dynamic programming

divide and conquer

greedy algorithms

# How hard are computational problems?

lower bounds on resources
(rule out efficient algorithms)

combinatorial arguments with computational models (decision trees, machines, circuits)

diagonalization

algebraic techniques (polynomials, tensors)

grouping problems into complexity classes

## **Which resources** can be analyzed?

- running time (poly, exp, sublinear)
- memory (e.g., RAM)
- parallelism (multiple CPUs, GPUs)
- randomness (vs. pseudorandomness)
- quantum entanglement
- **queries**

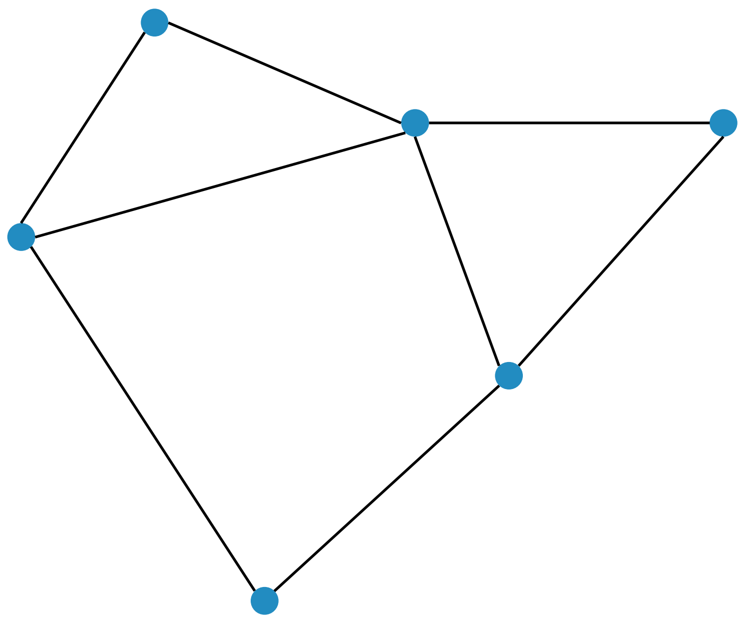
Graphs, Groups, Topology,
and Computational Complexity

# Graph isomorphism

graph
$G = (V, E)$

**vertex set**
$V = \{1, \ldots, n\}$

**edge set**
$E \subseteq \binom{V}{2}$



graph
$G = (V, E)$

**isomorphism**
bijection $f: V \to V'$ with
$uv \in E \Leftrightarrow f(u)f(v) \in E'$

graph
$G' = (V', E')$

**subgraph isomorphism**
injective function $g : V \to V'$ with
$uv \in E \Rightarrow g(u)g(v) \in E'$

*„Here are $G, G'$. Is there such $g$?“*

- **NP-hard**

*„Is there isomorphism $f$?“*

- **probably not NP-hard**
- **actually in $n^{\log^{O(1)} n}$ time**

graph
$G = (V, E)$

**isomorphism**
bijection $f: V \to V'$ with
$uv \in E \iff f(u)f(v) \in E'$

graph
$G' = (V', E')$

Luks 1981: **Isomorphism of Graphs of Bounded Valence can be Tested in Polynomial Time.**

Arvind et al. 2015: **Colored Hypergraph Isomorphism is Fixed Parameter Tractable.**

Seress 2003: **Permutation Group Algorithms**

*„Is there isomorphism $f$?“*

- **probably not NP-hard**
- **actually in $n^{\log^{O(1)} n}$ time**

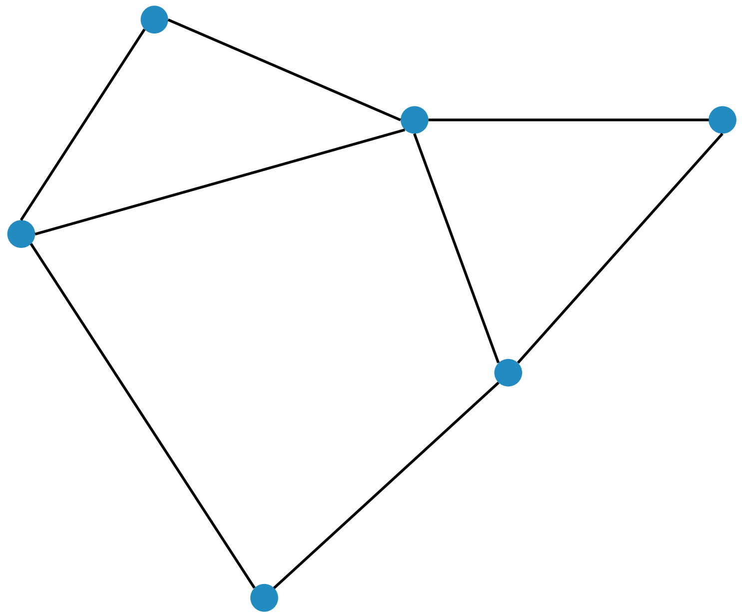
Graphs, Groups, Topology,
and Computational Complexity

## Evasiveness conjecture

# graph
$G = (V, E)$

**vertex set**
$V = \{1, \ldots, n\}$

**edge set**
$E \subseteq \binom{V}{2}$



$\mathcal{G}_n$ = set of graphs on $n$ vertices

# graph property $\Phi$

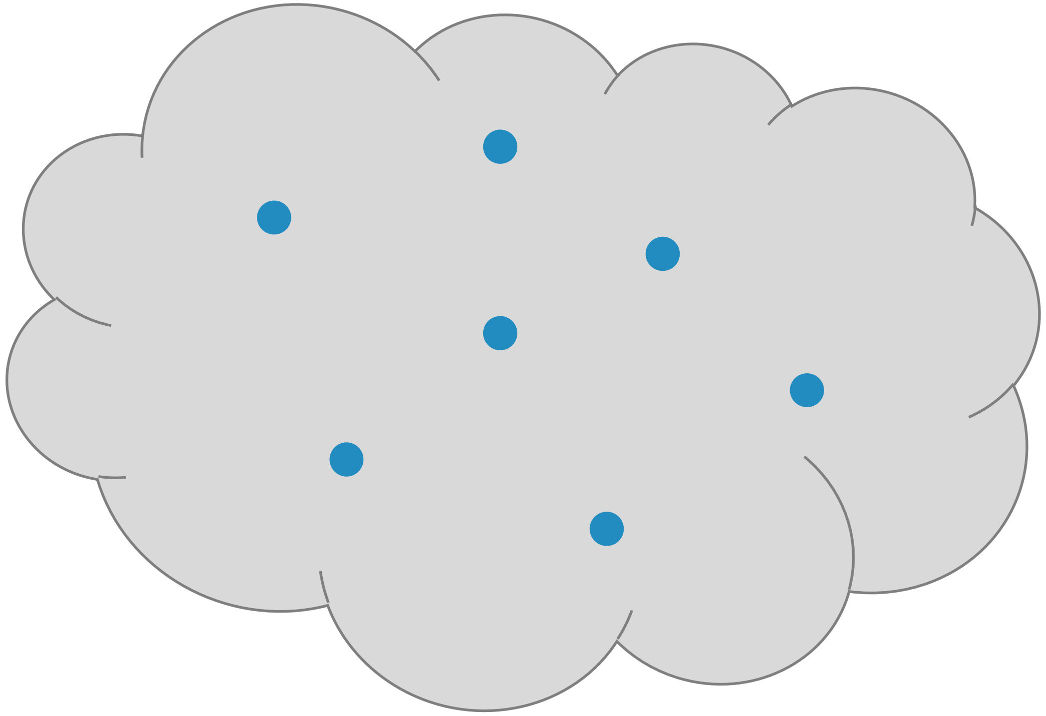
$\Phi : \mathcal{G}_n \to \{0,1\}$

invariant under isomorphisms:
$\Phi(G) = \Phi(G')$ if $G, G'$ isomorphic

graph
$G = (V, E)$

**vertex set**
$V = \{1, \dots, n\}$

**edge set**
$E \subseteq \binom{V}{2}$



$\mathcal{G}_n$ = set of graphs on $n$ vertices

graph property $\Phi$

$\Phi : \mathcal{G}_n \to \{0,1\}$

invariant under isomorphisms:
$\Phi(G) = \Phi(G')$ if $G, G'$ isomorphic

**Query complexity of $\Phi$:**
How many edges do we need to uncover **in the worst case** to know $\Phi(G)$ certainly?

**Evasiveness Conjecture:**
For every monotone $\Phi$,
all edges need to be uncovered!

shown when n is a prime power

Kahn, Saks, Sturtevant 1984:
**A topological approach to evasiveness.**

Miller 2013: **Evasiveness of Graph Properties and Topological Fixed-Point Theorems**

Löh 2022: **Applied Algebraic Topology**

Roth, Schmitt 2020:
**A topological approach to #W[1]-hardness.**

C., Neuen (unpublished):
**Counting Small Induced Subgraphs: Hardness via Fourier Analysis**

$\mathcal{G}_n$ = set of graphs on $n$ vertices

# graph property $\Phi$

$\Phi : \mathcal{G}_n \to \{0,1\}$

invariant under isomorphisms:
$\Phi(G) = \Phi(G')$ if $G, G'$ isomorphic

**Query complexity of $\Phi$:**
How many edges do we need to uncover **in the worst case** to know $\Phi(G)$ certainly?

**Evasiveness Conjecture:**
For every monotone $\Phi$,
all edges need to be uncovered!

shown when $n$ is a prime power

# Administrative fine-print

**Studienleistung (course work)**

- 80 minutes talk
  + 10 minutes discussion
- handout (in English):
  - 1-2 pages
  - talk summary
  - exercises for participation
- discussion of preliminary version 2-3 weeks before talk

**Prüfungsleistung (examination, graded)**

- written report of the talk, due one week before the talk
- Proseminar / ungraded seminar: report not mandatory but highly recommended
- precise ECTS depends on Prüfungsordnung & Modulkatalog

LaTeX templates for handout & report provided

**research**

- start early
- understand material
- consult additional sources, be critical and independent

**raw concept**

- collect important points
- find main goal of presentation, determine priorities
- devise rough structure

**detailed concept**

- be precise and concise
- order material
- ideas > technical details
- include questions
- end with summary, example, open problem
- practice timing
- seminar context

**report**

- more detailed than talk
- your account of material, cite properly, no plagiarism
- statements require proofs

**presentation**

- memorize first sentences
- be independent of notes, keep eye contact
- meta-information
- definitions on board, handout
- no rush, write clearly, use colors
- no need to impress anyone
- Can you follow your own talk?