Group Cohomology – Exercises

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Exercise 1 (a trace on group rings?). Let G be a group and let

$$\begin{aligned} \tau \colon \mathbb{Z}G &\longrightarrow \mathbb{Z} \\ \sum_{g \in G} a_g \cdot g &\longmapsto a_e. \end{aligned}$$

Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. For all $a, b \in \mathbb{Z}G$, we have $\tau(a \cdot b) = \tau(b \cdot a)$.
2. For all $a \in \mathbb{Z}G$, we have $\tau(a \cdot a^*) \geq 0$, where $(\sum_{g \in G} a_g \cdot g)^* := \sum_{g \in G} a_{g^{-1}} \cdot g$.

Exercise 2 (standard (co)chain complexes in the literature).

1. What is "$C^q(Q, G)$" from the following article in our notation? [I on p. 3/4]

   S. Eilenberg. Topological methods in abstract algebra. Cohomology theory of groups, *Bull. Amer. Math. Soc.*, 55, pp. 3–37, 1949.
2. What is the "bar resolution $C_*(\Gamma)$" from the following article in our notation? [Definition and Lemma 2.1 a)]

   M. Puschnigg. The Kadison-Kaplansky conjecture for word-hyperbolic groups, *Invent. Math.*, 149(1), pp. 153–194, 2002.

Hints. Of course, you do *not* need to read/understand the whole article. It suffices to untangle the terminology and to compare it to our setup. You have to justify your answer in your submission (e.g., by an explicit comparison).

Exercise 3 (the augmentation ideal). Let G be a group and let $I(G) := \ker \varepsilon$ be the *augmentation ideal* (where $\varepsilon \colon C_0(G) = \mathbb{Z}G \longrightarrow \mathbb{Z}$ is the augmentation map).

1. Show that $I(G) = \operatorname{Span}_{\mathbb{Z}}\{g - 1 \mid g \in G\}$.
2. Show that $I(G) = \operatorname{Span}_{\mathbb{Z}G}\{s - 1 \mid s \in S\}$ holds for every generating set $S \subset G$ of G.

Exercise 4 (group rings of cyclic groups). Let $n \in \mathbb{N}_{>0}$, let $G := \mathbb{Z}/n$, let $t := [1] \in G$, and let $N := \sum_{j=0}^{n-1} t^j \in \mathbb{Z}G$. For $a \in \mathbb{Z}G$, we consider the associated $\mathbb{Z}G$-homomorphism $M_a \colon \mathbb{Z}G \longrightarrow \mathbb{Z}G$ given by right multiplication with a.

1. Show that $\operatorname{im} M_N = \ker M_{t-1}$.
2. Show that $\operatorname{im} M_{t-1} = \ker M_N$.

Bonus problem (Kaplansky zero divisor conjecture and unique products).

1. What is the *unique product* property of groups?
2. Give an example of a (non-trivial) group with the unique product property and an example of a group without the unique product property.
3. Show that the group ring $\mathbb{Z}G$ has no non-trivial zero divisors if G is a group with the unique product property.

Submission before May 6, 2019, 10:00, in the mailbox

(Solutions may be submitted in English or German.)