Group Cohomology – Exercises

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Exercise 1 (Tor)**.** In the following, all modules carry the trivial group action. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. $\operatorname{Tor}_2^{\mathbb{Z}[\mathbb{Z}^2]}(\mathbb{Z}, \mathbb{Z}) \cong_{\mathbb{Z}} 0$
2. $\operatorname{Tor}_1^{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}^2, \mathbb{Z}^2) \cong_{\mathbb{Z}} 0$

Exercise 2 (algebraic mapping cones)**.** Let R be a ring and let $f_*\colon C_* \longrightarrow D_*$ be a chain map of R-chain complexes. The *mapping cone of* f_* is the R-chain complex $\operatorname{Cone}_*(f_*)$ consisting of the chain modules

$$\operatorname{Cone}_n(f_*) := C_{n-1} \oplus D_n$$

for all $n \in \mathbb{N}$ (where $C_{-1} := 0$), equipped with the boundary operators

$$\begin{aligned} \partial_n\colon \operatorname{Cone}_n(f_*) &\longrightarrow \operatorname{Cone}_{n-1}(f_*) \\ (x,y) &\longmapsto \bigl(-\partial^C_{n-1}(x), \partial^D_n(y) - f_{n-1}(x)\bigr) \end{aligned}$$

for all $n \in \mathbb{N}_{>0}$. Show the mapping cone trick, i.e., that $f_*\colon C_* \longrightarrow D_*$ is a quasi-isomorphism if and only if

$$\forall_{n\in\mathbb{N}} \quad H_n\bigl(\operatorname{Cone}(f_*)\bigr) \cong_R 0.$$

Hints. For the boundary operator on $\operatorname{Cone}(f_*)$, several different sign conventions are in use. Therefore, literature has to be used with care!

Exercise 3 (quasi-isomorphisms of complexes of projectives)**.** Let R be a ring. Prove that if C_* and D_* are ($\mathbb{N}$-indexed) R-chain complexes that consist of projective R-modules, then every quasi-isomorphism $C_* \longrightarrow D_*$ is a chain homotopy equivalence.
Hints. Mapping cone . . .

Exercise 4 (injectivity)**.** We consider the article

> N.V. Ivanov. Foundations of the theory of bounded cohomology, *J. Soviet Math.*, 37, pp. 1090–1114, 1987.

1. How are ordinary *injective modules* defined in module categories?
2. How are *relatively injective Banach G-modules* defined in the article?
3. What is the fundamental theorem of homological algebra in this context?
4. Use MathSciNet (https://www.ams.org/mathscinet) to find an article that solves the problem in Remark 3.9.1.

   Hints. Use the "Citations" tool!

Bonus problem (mapping cones in algebra and topology)**.**

1. How can one relate algebraic mapping cones to topological mapping cones?
2. What are differences/similarities between the properties of algebraic mapping cones and topological mapping cones, respectively?

Submission before July 8, 2019, 10:00, in the mailbox