Group Cohomology – Exercises

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Exercise 1 (induced maps in group homology)**.** Let $\varphi\colon G \longrightarrow H$ be a group homomorphism. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If φ is injective, then $H_1(\varphi;\mathbb{Z})\colon H_1(G;\mathbb{Z}) \longrightarrow H_1(H;\mathbb{Z})$ is injective.
2. If φ is surjective, then $H_1(\varphi;\mathbb{Z})\colon H_1(G;\mathbb{Z}) \longrightarrow H_1(H;\mathbb{Z})$ is surjective.

Exercise 2 (finitary symmetric groups)**.** Let X be a set with $|X| \geq 2$ and let $\operatorname{FSym}(X)$ be the group of all finitary permutations of X; a bijection $f\colon X \longrightarrow X$ is *finitary* if the set $\{x \in X \mid f(x) \neq x\}$ is finite.

1. Compute $H^1\bigl(\operatorname{FSym}(X);\mathbb{Z}\bigr)$.
2. Compute $H_1\bigl(\operatorname{FSym}(X);\mathbb{Z}\bigr)$.

Exercise 3 (certain groups of homeomorphisms in the literature)**.** We consider the following article:

> J. N. Mather. The vanishing of the homology of certain groups of homeomorphisms, *Topology*, 10, pp. 297–298, 1971.

1. What is the main result of this article?
2. Name at least two further published articles whose titles contain the string "certain groups of homeomorphisms".

   Hints. The database https://mathscinet.ams.org/mathscinet might help. Access to Mathscinet requires a subscription; the website can be accessed through the campus network (or using SSH tunnels via UR).
3. *Bonus problem.* Can you make sense of the definition of $C(G)$ in the proof of the lemma on p. 297?
4. *Bonus problem.* What is φ in (a) on p. 298? How do you think that this happened?

Exercise 4 (the integral Heisenberg group)**.** Let

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\} \subset \operatorname{SL}(3, \mathbb{Z})$$

be the *(integral) Heisenberg group.*

1. Compute $H_1(H;\mathbb{Z})$.
2. Show that $\operatorname{rk} H = 2$.

Please turn over

Bonus problem (a perfect homeomorphism group)**.** Let $n \in \mathbb{N}_{>0}$. For a homeomorphism $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we define the *support* by

$$\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n \mid f(x) \neq x\}} \subset \mathbb{R}^n.$$

We say that f has *compact support* if $\operatorname{supp} f$ is compact. Let G be the group(!) of all homeomorphisms $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ with compact support. Show that G is perfect (whence $H_1(G;\mathbb{Z}) \cong_{\mathbb{Z}} 0$).

Hints. Let $f \in G$, let B be an open ball containing $\operatorname{supp} f$, and let $g \in G$ with $g^k(B) \cap g^m(B) = \emptyset$ for all $k, m \in \mathbb{N}$ with $k \neq m$ (why does such a g exist?). Then consider the following situation: