Group Cohomology – Exercises

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Exercise 1 (projectivity)**.** Let G be a group and let $\mathbb{Z}$ be equipped with the trivial G-action. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $\mathbb{Z}$ is a a projective $\mathbb{Z}G$-module, then G is finite.
2. If G is finite, then $\mathbb{Z}$ is a projective $\mathbb{Z}G$-module.

Exercise 2 (cohomology of $\mathbb{Z}/3$)**.** Use group extensions to show that $H^2(\mathbb{Z}/3;\mathbb{Z})$ contains at least three elements (where $\mathbb{Z}$ carries the trivial action). More precisely: Provide three extensions of $\mathbb{Z}/3$ by $\mathbb{Z}$ that are pairwise non-equivalent.

Exercise 3 (cohomology of $\mathbb{Z}^2$)**.** Use group extensions to show that $H^2(\mathbb{Z}^2;\mathbb{Z}) \not\cong_{\mathbb{Z}} 0$ (where $\mathbb{Z}$ carries the trivial action).
Hints. The Heisenberg group (Sheet 2, Exercise 4) can serve as a middleman.

Exercise 4 (extensions in the literature)**.** We consider the following article:

> M. Bucher, R. Frigerio, T. Hartnick. A note on semi-conjugacy for circle actions, *L'Enseignement Mathématique (2)*, 62, pp. 317–360, 2016.

1. What is "$e(\xi)$" from this article (paragraph before Lemma 3.1) in our notation? In particular, how does "c_σ" relate to our notation?
2. Give a proof of Lemma 3.1 (in our notation).

Bonus problem (cohomology of homeomorphisms on the circle)**.** We consider the circle $S^1 := \mathbb{R}/\mathbb{Z}$, the group $G := \operatorname{Homeo}^+(S^1)$ of orientation-preserving homeomorphisms of S^1, and the subgroup

$$\widetilde{G} := \bigl\{ f \in \operatorname{Homeo}^+(\mathbb{R}) \bigm| \forall_{x\in\mathbb{R}} \quad f(x+1) = f(x) + 1 \bigr\}$$

of the orientation-preserving homeomorphisms of $\mathbb{R}$. A homeomorphism $\mathbb{R} \longrightarrow \mathbb{R}$ is *orientation-preserving* if it is monotonically increasing. Moreover, a homeomorphism $S^1 \longrightarrow S^1$ is *orientation-preserving* if it preserves orientations in the sense of linear algebra (which can be defined via suitable determinants). Let $p\colon \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} = S^1$ be the projection map and let

$$\begin{aligned} \pi\colon \widetilde{G} &\longrightarrow G \\ f &\longmapsto \bigl([x] \mapsto p(f(x))\bigr) \end{aligned}$$

1. Show that π is a well-defined group homomorphism and that there is a central extension of the form

   $$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{G} \stackrel{\pi}{\longrightarrow} G \longrightarrow 1.$$

   Hints. For topologists, surjectivity of π should be easy!
2. Show that this extension is *not* trivial and conclude that $H^2(G;\mathbb{Z}) \not\cong_{\mathbb{Z}} 0$.
   Hints. Torsion!

Submission before May 20, 2019, 10:00, in the mailbox