Group Cohomology – Exercises

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Exercise 1 ((co)homology with group ring coefficients)**.** Let $G := \mathbb{Z}$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. We have $H_1(G; \mathbb{Z}G) \cong_{\mathbb{Z}} 0$.
2. We have $H^1(G; \mathbb{Z}G) \cong_{\mathbb{Z}} 0$.

Exercise 2 ((co)homology of $\mathbb{Z}^2$)**.** Let $T := \mathbb{Z}^2$.

1. Show that $\mathbb{Z}T \cong_{\mathsf{Ring}} \mathbb{Z}[a,b]_S$, where $S := \{a^n \cdot b^m \mid n, m \in \mathbb{N}\}$.
2. Show that the complex $\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}T \xrightarrow{\partial_2} \mathbb{Z}T \oplus \mathbb{Z}T \xrightarrow{\partial_1} \mathbb{Z}T$, where

$$\begin{aligned} \partial_2 \colon \mathbb{Z}T &\longrightarrow \mathbb{Z}T \oplus \mathbb{Z}T \\ x &\longmapsto \bigl(x \cdot (1-b), x \cdot (a-1)\bigr) \\ \partial_1 \colon \mathbb{Z}T \oplus \mathbb{Z}T &\longrightarrow \mathbb{Z}T \\ (x,y) &\longmapsto x \cdot (a-1) + y \cdot (b-1), \end{aligned}$$

   together with the augmentation $\varepsilon \colon \mathbb{Z}T \longrightarrow \mathbb{Z}$ is a projective resolution of the trivial $\mathbb{Z}T$-module $\mathbb{Z}$ over $\mathbb{Z}T$.
3. Compute $H_*(T;\mathbb{Z})$ and $H^*(T;\mathbb{Z})$ (with the trivial T-action on $\mathbb{Z}$).
4. Compute $H^1(T;\mathbb{Z}T)$.
5. *Bonus problem.* What could be the geometric background of the above projective resolution?

Exercise 3 (cohomology of free groups)**.** Let F be a free group of rank 2. Show that $H^1(F;\mathbb{Z}F)$ is *not* a finitely generated $\mathbb{Z}$-module.
Hints. For a free generating set $\{a,b\}$ of F, it might help to consider $(b^n, a^n) \in \mathbb{Z}F \times \mathbb{Z}F$, the reduction to $\mathbb{Z}[F_{\mathrm{ab}}] \times \mathbb{Z}[F_{\mathrm{ab}}]$, and the resolution from Exercise 2.

Exercise 4 (Shapiro lemma in the literature)**.** We consider the following article:

> R.G. Swan. Groups of cohomological dimension one, *Journal of Algebra*, 12, pp. 585–601, 1969.

1. How is $\mathrm{cd}_{\mathbb{Z}}$ defined and what does the Shapiro lemma say about $\mathrm{cd}_{\mathbb{Z}}$?
2. Which result does Swan use to derive Theorem B from Theorem A?

Bonus problem (some congruences for $\mathbb{Z}/p$-actions on $\mathbb{Z}/p^n$)**.** Let $p \in \mathbb{N}$ be an odd prime, let $n \in \mathbb{N}_{>1}$, and let $a \in \mathbb{Z}$ with $a^p \equiv 1 \mod p^n$.

1. Show that $a \equiv 1 \mod p^{n-1}$.
2. Conclude that if $a \not\equiv 1 \mod p^n$, there exists a $k \in \{1, \dots, p-1\}$ with $a^k \equiv 1 + p^{n-1} \mod p^n$.

Submission before May 27, 2019, 10:00, in the mailbox