Group Cohomology – Exercises

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Exercise 1 (homology of dihedral groups)**.** For $n \in \mathbb{N}_{\geq 3}$, let D_n denote the *dihedral group for n* (i.e., the isometry group of a regular Euclidean n-gon). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. $H_{2019}(D_{2021}; \mathbb{Z}/2020) \cong_{\mathbb{Z}} 0$
2. $H_{2019}(D_{2020}; \mathbb{Z}/2021) \cong_{\mathbb{Z}} 0$

Hints. You may use the description $D_n \cong_{\mathsf{Group}} \mathbb{Z}/n \rtimes \mathbb{Z}/2$, where $[1] \in \mathbb{Z}/2$ acts by (additive) inversion on $\mathbb{Z}/n$ (Proposition III.1.1.57).

Exercise 2 (the infinite dihedral group)**.** The *infinite dihedral group D_∞* is the isometry group of the metric space $\mathbb{Z}$ with respect to the metric inherited from the standard metric on $\mathbb{R}$. Let t denote the reflection at 0, let s denote the translation by 1, and let t' denote the reflection at $1/2 \in \mathbb{R}$. Solve two out of the following four problems:

1. Show that $S := \{s, t\}$ is a generating set of D_∞ and that D_∞ is isomorphic to a suitable semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}/2$.
2. Show that the word metric on D_∞ associated with S is isometric to the word metric on $\mathbb{Z} \times \mathbb{Z}/2$ associated with the generating set $\{(1, 0), (0, [1])\}$.
3. Show that $T := \{t, t'\}$ is a generating set of D_∞.
4. Show that the word metric on D_∞ associated with T is isometric to the word metric on $\mathbb{Z}$ associated with the generating set $\{1\}$.

Exercise 3 (metric embedding notions in the literature)**.** We consider:

> J. Block, S. Weinberger. Aperiodic tilings, positive scalar curvature, and amenability of spaces, *J. Amer. Math. Soc.*, 5(4), pp. 907–918, 1992.

1. Prove the statement "Also note that a coarse quasi-isometry in the sense of Gromov is an EPL map." (p. 909). More precisely: Show that every quasi-isometric embedding between metric spaces is an effectively proper Lipschitz map (defined on p. 909).
2. Does the converse also hold? Justify your answer!

Exercise 4 ((co)induction of finite index subgroups)**.** Let G be a group and let $H \subset G$ be a subgroup of finite index. We consider

$$\begin{aligned}
\varphi\colon \operatorname{Ind}_H^G(B) = \mathbb{Z}G \otimes_{\mathbb{Z}H} B &\longrightarrow \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B) = \operatorname{Coind}_H^G(B)\\
g \otimes b &\longmapsto \bigl(x \mapsto \chi_H(x \cdot g) \cdot (x \cdot g) \cdot b\bigr)\\
\psi\colon \operatorname{Coind}_H^G B = \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B) &\longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} B = \operatorname{Ind}_H^G(B)\\
f &\longmapsto \sum_{gH \in G/H} g \otimes f(g^{-1})
\end{aligned}$$

Show that φ and ψ are well-defined $\mathbb{Z}G$-linear maps and that φ and ψ are mutually inverse.

Please turn over

Bonus problem (Legendre symbol and transfer)**.** Let $p \in \mathbb{N}$ be an odd prime.

1. What is the *Legendre symbol* associated with p ?
2. Show that the Legendre symbol "coincides" with the transfer on $H_1(\,\cdot\,;\mathbb{Z})$ of the subgroup $\{-1,+1\}$ of the multiplicative group $\bigl(\mathbb{Z}/(p)\bigr)^{\times}$.

Submission before June 3, 2019, 10:00, in the mailbox