Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 6, June 3, 2019

Exercise 1 (UDBG spaces). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. Every uniformly discrete metric space has bounded geometry.
2. Every metric space with bounded geometry is uniformly discrete.

Exercise 2 (uniformly finite chains). Let R be a normed ring with unit, let (X, d) be a UDBG space, let $n \in \mathbb{N}_{\geq 1}$, let $c = \sum_{x \in X^{n+1}} c_x \cdot x \in C_n^{\text{uf}}(X; R)$, and let $j \in \{0, \dots, n\}$. Show that the following map is a well-defined chain in $C_{n-1}^{\text{uf}}(X; R)$:

$$\begin{aligned} X^n &\longrightarrow R \\ y &\longmapsto \sum_{x \in \{z \in X^{n+1} \mid (z_0, \dots, \widehat{z}_j, \dots, z_n) = y\}} c_x. \end{aligned}$$

Exercise 3 (the fundamental class in uniformly finite homology). Let $(X, d) := (\mathbb{Z}, d_{\{1\}})$ and for $A \subset X$ let $[A]_{\mathbb{Z}} \in H_0^{\text{uf}}(X; \mathbb{Z})$ be the homology class represented by the uniformly finite cycle $\sum_{x \in A} 1 \cdot x$.

1. Show that for every finite set $A \subset X$, we have $[A]_{\mathbb{Z}} = 0$ in $H_0^{\text{uf}}(X; \mathbb{Z})$.
2. Show that for each $n \in \mathbb{N}_{>0}$, there exists a class $\alpha_n \in H_0^{\text{uf}}(X; \mathbb{Z})$ that satisfies $n \cdot \alpha_n = [X]_{\mathbb{Z}}$ in $H_0^{\text{uf}}(X; \mathbb{Z})$.

Exercise 4 (means). We consider the following article:

> M. Gromov. Volume and bounded cohomology, *Publ. Math. IHES*, 56, pp. 5–99, 1982.

1. Where and how are amenable groups defined in this article?
2. Show that this notion of amenability is equivalent to ours; more precisely, show that an $\mathbb{R}$-linear map $m \colon \ell^\infty(G, \mathbb{R}) \longrightarrow \mathbb{R}$ is a left-invariant mean on G if and only if all of the following conditions are satisfied:
   - $m(1) = 1$
   - $m(g \cdot f) = m(f)$ for all $f \in \ell^\infty(G, \mathbb{R})$ and all $g \in G$
   - $\bigl|m(f)\bigr| \leq |f|_\infty$ for all $f \in \ell^\infty(G, \mathbb{R})$

Bonus problem (Ponzi schemes).

1. Who was Charles Ponzi?
2. What is a *Ponzi scheme*?

Submission before June 10, 2019, 10:00, in the mailbox

As June 10 is a holiday: extended deadline: June 11, 10:00