# Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig  Sheet 7, June 10, 2019

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**Exercise 1** (uniformly finite homology of groups). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. $H_0^{\text{uf}}\bigl(\langle a, b \mid a^{2019}, aba^{-1}b^{-1}\rangle; \mathbb{Z}\bigr) \cong_{\mathbb{Z}} 0$
2. $H_0^{\text{uf}}\bigl(\langle a, b, c, d \mid c^{2019}d^{2020}\rangle; \mathbb{Z}\bigr) \cong_{\mathbb{Z}} 0$

**Exercise 2** ($\mathbb{Z} \not\sim_{\text{QI}} \mathbb{Z}^2$). Use uniformly finite homology $H_*^{\text{uf}}(\,\cdot\,;\mathbb{R})$ to prove that $\mathbb{Z}$ and $\mathbb{Z}^2$ are *not* quasi-isometric.
*Hints.* Amenability calls for transfer! And Sheet 4 might help.

**Exercise 3** (quasi-isometry vs. bilipschitz equivalence). We consider the following article:

> K. Whyte. Amenability, bi-Lipschitz equivalence, and the von Neumann conjecture, *Duke Math. J.*, 99(1), pp. 93–112, 1999.

1. Give a proof of the statement “Observe that a quasi-isometry between UDBG spaces is bilipschitz if and only if it is bijective” at the beginning of the proof of Theorem 4.1.
2. How is this fact used in the proof of Theorem 4.1?

**Exercise 4** (homology of free groups with $\ell^2$-coefficients). Let $F$ be a free group of rank 2. Give an explicit example of a chain $b \in C_1(F; \ell^2(F, \mathbb{R}))$ (with the usual left $\mathbb{Z}F$-module structure on $\ell^2(F, \mathbb{R})$) that satisfies

$$\partial_1 b = e \otimes \chi_{\{e\}} \in C_0\bigl(F; \ell^2(F, \mathbb{R})\bigr).$$

Sketch this chain!

**Bonus problem** (measure equivalence).

1. What is *measure equivalence* of (countable) groups?
2. How does the definition of measure equivalence relate to the dynamical criterion for quasi-isometry of (finitely generated) groups?
3. How does measure equivalence relate to orbit equivalence of group actions?

   *Hints.* It is enough to cite such a result from the literature; no proof is required.
4. Give an example of (non-trivial) coefficients for group (co)homology that lead to measure equivalence invariants!

   *Hints.* It is enough to cite such a result from the literature; no proof is required.

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Submission before June 17, 2019, 10:00, in the mailbox