Group Cohomology – Exercises

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Exercise 1 (commutator length)**.** Let G be a group. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $g \in [G,G]$ and $h \in G$, then $\operatorname{cl}_G(h \cdot g \cdot h^{-1}) = \operatorname{cl}_G(g)$.
2. If $g, h \in [G,G]$, then $\operatorname{cl}_G(g \cdot h) = \operatorname{cl}_G g + \operatorname{cl}_G h$.

Exercise 2 (homogenisation)**.** Let G be a group and let $\varphi \colon G \longrightarrow \mathbb{R}$ be a quasi-morphism.

1. Show that the following map is a well-defined homogeneous quasi-morphism on G that is uniformly close to φ (this requires, in particular, a proof of the existence of the limit on the right-hand side):
$$\begin{aligned} \overline{\varphi} \colon G &\longrightarrow \mathbb{R} \\ g &\longmapsto \lim_{n \to \infty} \frac{1}{n} \cdot \varphi(g^n) \end{aligned}$$
2. Conclude that the following map is a well-defined isomorphism of $\mathbb{R}$-vector spaces:
$$\begin{aligned} \operatorname{QM}(G) / \operatorname{QM}_0(G) &\longrightarrow \overline{\operatorname{QM}}(G) / \operatorname{Hom}_{\mathsf{Group}}(G, \mathbb{R}) \\ [\varphi] &\longmapsto [\overline{\varphi}] \end{aligned}$$

Exercise 3 (rotation number)**.** We consider the following article:

> K. Mann. Rigidity and flexibility of group actions on the circle, *Handbook of group actions*, IV, pp. 705–752, *Adv. Lect. Math.*, 41, International Press, 2018.
> The preprint version is freely available at: https://arxiv.org/abs/1510.00728

Let $\widetilde{G} := \{f \in \operatorname{Homeo}^+(\mathbb{R}) \mid \forall_{x \in \mathbb{R}} \quad f(x+1) = f(x) + 1\}$ be the group (with respect to composition) of periodic orientation-preserving (i.e., monotonically increasing) homeomorphisms of $\mathbb{R}$ (see also the bonus exercise on Sheet 3).

1. How is the *rotation number* $\widetilde{\operatorname{rot}} \colon \widetilde{G} \longrightarrow \mathbb{R}$ on $\widetilde{G}$ defined? Why is this definition well-defined? (You should give more details than the article ...)
2. Show that $\widetilde{\operatorname{rot}}$ is a homogeneous quasi-morphism on $\widetilde{G}$.

Exercise 4 (means from vanishing bounded cohomology)**.** Let G be a group, let $V := \ell^\infty(G, \mathbb{R})/C$, where $C \subset \ell^\infty(G, \mathbb{R})$ is the subspace of constant functions, and suppose that $H^1_{\mathrm{b}}(G; V^\#) \cong_{\mathbb{R}} 0$. Show that there exists a bounded $\mathbb{R}$-linear functional $\mu \colon \ell^\infty(G, \mathbb{R}) \longrightarrow \mathbb{R}$ with $\mu(1) = 1$ that is left-invariant.

Bonus problem (acronyms)**.** How do the following acronyms expand in Mathematics? Where are they located?

MFO, MSRI, RIMS, IAS, BIRS, IMPAN,
IML, MPIM, INI, IHÉS, PIMS, KIAS

Submission before June 24, 2019, 10:00, in the mailbox