Group Cohomology – Exercises

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Exercise 1 (second cohomology and stable commutator length)**.** Let G be a group. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $H^2_b(G;\mathbb{R}) \cong_\mathbb{R} 0$, then $\operatorname{scl}_G = 0$.
2. If $H^2(G;\mathbb{R}) \cong_\mathbb{R} 0$, then $\operatorname{scl}_G = 0$.

Exercise 2 (torsion groups)**.** Let G be a torsion group (i.e., every element in G has finite order).

1. Compute the function $\operatorname{scl}_G \colon [G,G] \longrightarrow \mathbb{R}$ and the space $\overline{\mathrm{QM}}(G)$ (directly, without using Bavard duality).
2. Let G, in addition, have the property that every group extension of the form $0 \longrightarrow \mathbb{R} \longrightarrow ? \longrightarrow G \longrightarrow 1$ splits. Show that $H^2_b(G;\mathbb{R}) \cong_\mathbb{R} 0$.

Exercise 3 (quasi-isomorphisms)**.** Let R be a ring. A chain map $f_* \colon C_* \longrightarrow D_*$ in ${}_R\mathsf{Ch}$ is a *quasi-isomorphism* if, for each $n \in \mathbb{N}$, the induced homomorphism $H_n(f_*) \colon H_n(C_*) \longrightarrow H_n(D_*)$ is an isomorphism. Show (via an example over a suitable ring R) that if there exists a quasi-isomorphism $C_* \longrightarrow D_*$, then, in general, there is *no* quasi-isomorphism $D_* \longrightarrow C_*$.

Exercise 4 (exact categories)**.** We consider the article

T. Bühler. Exact categories, *Expo. Math.*, 28, pp. 1–69, 2010.

Before proceeding, you should look up what an *additive category* is.

1. On p. 4, *admissible epics* are defined. Make this definition explicit.
2. What is the *obscure axiom*?
3. Why is it called obscure?
4. How are *exact functors* between exact categories defined?

Bonus problem (duality principle for semi-norms on homology)**.** Let C_* be a chain complex in the category of normed $\mathbb{R}$-vector spaces (and bounded linear operators) and let $D^* := \operatorname{BHom}(C_*, \mathbb{R})$ be the dual cochain complex. Let $n \in \mathbb{N}$ and let $\alpha \in H_n(C_*)$ be represented by the cycle $c \in C_n$.

1. Show that
$$\|\alpha\| = \sup\Bigl\{ \frac{1}{|f|_\infty} \Bigm| f \in D^n,\ \delta^n f = 0,\ f(c) = 1 \Bigr\}.$$
Here, $\|\cdot\|$ denotes the semi-norm on $H_n(C_*)$ induced by the norm on C_n and δ^* is the coboundary operator of D^*. Moreover, we set $\sup \emptyset := 0$.

   Hints. Hahn-Banach!

2. Does there exist an amenable group G and a class $\alpha \in H_{2019}(G;\mathbb{R})$ with $\|\alpha\|_1 = 2019$? Here, $\|\cdot\|_1$ denotes the semi-norm on $H_*(G;\mathbb{R})$ induced by the ℓ^1-norm on $C^\mathbb{R}_*(G)$.

Submission before July 1, 2019, 10:00, in the mailbox